Fall 2008-2009 | **Final Exam** | **Date: January 30**
Prof: M. Egeileh | **Math 201 - Sections 17 to 20** | **Duration: 2 hours**

Exercise 1 *(5 pts)* Is the series $\sum_{n\geqslant 0} \frac{(-1)^n}{n!}$ convergent ? absolutely convergent ? Justify your answers.

Exercise 2 *(5 pts)* Does $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2-y^2}$ exist ? Justify your answer.

Exercise 3 *(17 pts)* We consider the function defined on $\mathbb{R}^2$ by : $f(x,y) = x^2 + 2y^2 - \frac{y^3}{3}$.

1. Find all the critical points of f. *(4 pts)*
2. Give the nature (local minimum, local maximum or saddle point) of each of the critical points you found in question 1. *(8 pts)*
3. Give an equation for the tangent plane to the graph of f at the point $(0, 4, f(0, 4))$. *(5 pts)*

Exercise 4 *(35 pts)* Let R be the region in the plane bounded by the triangle of vertices $O(0,0)$, $A(2,0)$ and $B(1,1)$. We denote by C_1 the line segment joining O to A, C_2 the line segment joining A to B, and C_3 the line segment joining B to O.

1. (a) Compute $\iint_R (y - x^2)\, dx\, dy$, using the order $dx\, dy$. *(6 pts)*
   (b) Write the above double integral as iterated integrals with the order $dy\, dx$ (you do **not** have to re-calculate its value). *(4 pts)*

2. Let $\vec{F}$ be the vector field defined on $\mathbb{R}^2$ by : $\vec{F}(x,y) = (x + x^2 y)\vec{i} + (xy + y)\vec{j}$.
   (a) Find, by direct computation, the value of $\int_{C_3} \vec{F} \cdot \vec{dl}$ (work of $\vec{F}$ along C_3). *(6 pts)*
   (b) Compute curl $\vec{F}(x,y)$ (where curl $\vec{F}$ is the $\vec{k}$-component of the vector field $\overrightarrow{\text{curl}}\ \vec{F}$). *(2 pts)*
   (c) Apply Green's theorem to find the value of $\int_{C_1} \vec{F} \cdot \vec{dl} + \int_{C_2} \vec{F} \cdot \vec{dl}$. *(5 pts)*

3. Let $\vec{G}$ be the vector field defined on $\mathbb{R}^2$ by : $\vec{G}(x,y) = (2x)\vec{i} + (4y - y^2)\vec{j}$.
   (a) Show that $\vec{G}$ is conservative. *(4 pts)*
   (b) Give a potential function for $\vec{G}$. *(4 pts)*
   (c) What is the value of $\int_{C_1} \vec{G} \cdot \vec{dl} + \int_{C_2} \vec{G} \cdot \vec{dl}$? *(4 pts)*

Exercise 5 *(15 pts)* Compute the volume of the region D of the space lying in the first octant, bounded from below by the xy-plane, from the sides by the planes $x = 2$ and $y = 2$, and from above by the plane $x + y + 2z = 6$. The region D is sketched in page 2.

Exercise 6 *(12 pts)*

1. For each the following two power series, give the radius of convergence. It is sufficient to provide **justification for only one** of them. *(2 pts)*

   (a) $\sum_{n\geqslant 0} \frac{x^{2n}}{(2n)!}$

   (b) $\sum_{n\geqslant 0} \frac{x^{2n+1}}{(2n+1)!}$

2. For every $x \in \mathbb{R}$, we set $\text{ch}(x) = \frac{1}{2}(e^x + e^{-x})$ and $\text{sh}(x) = \frac{1}{2}(e^x - e^{-x})$. Find the Maclaurin series generated by each of the functions ch and sh. *(6 pts)*

3. Show that for any $t \in [0,1]$, $\text{ch}(t) \leqslant \frac{5}{3}$. (*Hint : you may use the fact that ch is an increasing function on* $[0,+\infty[$, *and that* $1 \leqslant ln(3)$). *(2 pts)*

4. Deduce from the preceding question an upper bound for the error committed when approximating $\frac{1}{2}(e+\frac{1}{e})$ by 1.5. You may leave the final answer as a fraction $\frac{a}{b}$. (*Hint : start by writing that* $ch(x) = 1 + \frac{x^2}{2!} + 0\frac{x^3}{3!} + R_3(x)$, *then use Taylor remainder's estimation theorem...*). *(2 pts)*

Exercise 7 *(11 pts)* Let R be the region inside the reversed empty ice-cream cone $z = -\sqrt{x^2+y^2}$ and sandwiched between the planes $z = -1$ and $z = -2$.

1. Find the volume of R using a triple integral in spherical coordinates, with the order $d\rho\, d\phi\, d\theta$. *(9 pts)*

2. Setup the triple integral whose evaluation would give you the volume of R, in spherical coordinates, with the order $d\phi\, d\rho\, d\theta$ (do **not** re-calculate the volume of R). *(2 pts)*

Math 201 — Fall 2009–10
Calculus and Analytic Geometry III, sections 1–8, 24–26
Final Exam, February 1 — Duration: 2.5 hours

GRADES:

1 (15 pts)	2 (12 pts)	3 (10 pts)	4 (12 pts)	5 (14 pts)	6 (12 pts)	7 (15 pts)
8 (15 pts)	9 (25 pts)	10 (28 pts)	11 (14 pts)	12 (18 pts)	13 (10 pts)	Total/200

YOUR NAME: Correction

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 1 Lecture MWF 3 Professor Makdisi Recitation F 11 Ms. Nassif	Section 2 Lecture MWF 3 Professor Makdisi Recitation F 2 Ms. Nassif	Section 3 Lecture MWF 3 Professor Makdisi Recitation F 4 Ms. Nassif	Section 4 Lecture MWF 3 Professor Makdisi Recitation F 9 Ms. Nassif
Section 5 Lecture MWF 10 Professor Raji Recitation T 11 Professor Raji	Section 6 Lecture MWF 10 Professor Raji Recitation T 3:30 Ms. Itani	Section 7 Lecture MWF 10 Professor Raji Recitation T 8 Ms. Itani	Section 8 Lecture MWF 10 Professor Raji Recitation T 2 Ms. Itani
Section 24 Lecture MWF 2 Professor Tlas Recitation F 11 Dr. Yamani	Section 25 Lecture MWF 2 Professor Tlas Recitation F 12 Dr. Yamani	Section 26 Lecture MWF 2 Professor Tlas Recitation F 3 Professor Tlas	

INSTRUCTIONS:

1. Write your NAME and AUB ID number, and circle your SECTION above.
2. Solve the problems inside the booklet. Explain your steps precisely and clearly to ensure full credit. Partial solutions will receive partial credit.
3. You may use the back of each page for scratchwork OR for solutions. There are five blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, INDICATE CLEARLY WHERE THE GRADER SHOULD CONTINUE READING.
4. Closed book and notes. NO CALCULATORS ALLOWED. Turn OFF and put away any cell phones.

GOOD LUCK!

An overview of the exam problems. Take a minute to look at all the questions, THEN solve each problem on its corresponding page INSIDE the booklet.

1. (5 pts each part, 15 pts total) Determine whether each of the following series converges or diverges:

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{\sqrt{n^5+1}}$ (b) $\sum_{n=1}^{\infty} \tan^{-1} n$ (c) $\sum_{n=1}^{\infty} (n+1)(0.5)^n$

2. (12 pts) Find the interval of convergence of the following series. Remember to test the endpoints.

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{\sqrt{n}\,3^n}$$

3. (5 pts each part, 10 pts total)
a) Find the first four nonzero terms of the Taylor series for e^x and $\cos 3x$ centered at $a = 0$.
b) Compute the limit: $\lim_{x \to 0} \frac{e^x - 1 - x}{1 - \cos 3x}$.

4. (6 pts each part, 12 pts total)
a) Find the second-order Taylor polynomial of the function $f(x) = \sqrt{x}$, **centered at** $x = 100$.
b) Using your answer P_2 above, give an approximate value for $\sqrt{99}$ and estimate the error.

5. (7 pts each part, 14 pts total) We are given a function $f(x, y)$ satisfying:

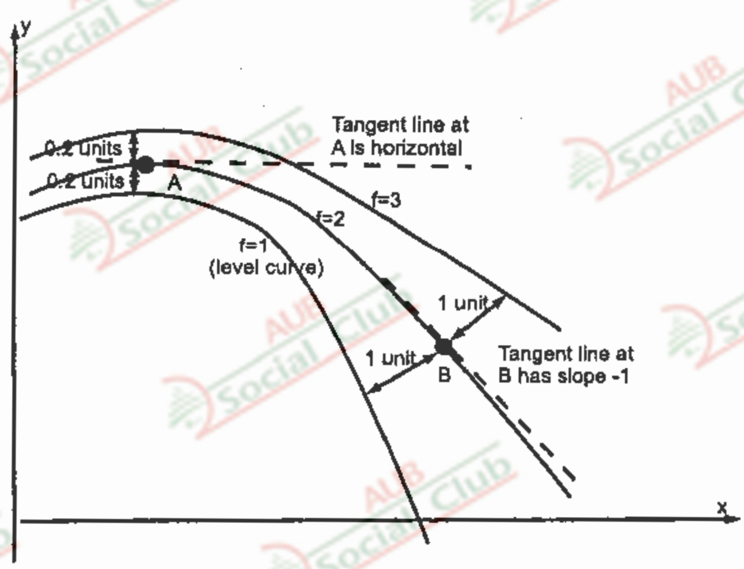
$$\vec{\nabla} f \Big|_{(1,2,3)} = (4, -5, 6), \qquad \vec{\nabla} f \Big|_{(2,4,6)} = (8, -9, 10), \qquad \vec{\nabla} f \Big|_{(2,0,1)} = (2, 0, 2).$$

a) Which of the following differences do you expect to be largest, and why?

$$f(1.1, 2.1, 3.1) - f(1, 2, 3), \qquad f(2.1, 4.1, 6.1) - f(2, 4, 6), \qquad f(2.1, 0.1, 1.1) - f(2, 0, 1).$$

b) Let $g(r, s) = f(r - s, r + s, 2rs)$. Find $\frac{\partial g}{\partial s}\Big|_{(r,s)=(3,1)}$.

6. (4 pts each part, 12 pts total) Some level curves of a function $f(x, y)$ are drawn below. **The rest of problem 6 is on page iii.**



6, continued. a) Give two **unit** vectors $\vec{u}_A, \vec{u}_B$ such that $\vec{u}_A$ points in the same direction as $\vec{\nabla} f \Big|_A$, and similarly $\vec{u}_B$ points in the same direction as $\vec{\nabla} f \Big|_B$.

b) Based on the figure, what is a reasonable approximation to the directional derivatives

$$D_{\vec{u}_A} f \Big|_A, \qquad D_{\vec{u}_B} f \Big|_B ?$$

c) Deduce a reasonable approximation to each of the vectors $\vec{\nabla} f \Big|_A$ and $\vec{\nabla} f \Big|_B$.

7. (15 pts) Find and classify the critical points of the function $f(x,y) = \frac{x^4}{12} - 3xy + \frac{3y^2}{2}$.

8. (15 pts total) Let $f(x,y,z) = 8x + y + z$, and consider the surface S given by $xyz = 1$ in the first octant only:

$$S = \{(x,y,z) \mid xyz = 1,\ x,y,z > 0\}.$$

a) (12 pts) Use Lagrange multipliers to find the minimum value of f on the surface S.

b) (3 pts) Show that f does not have a maximum value on S. (Why are the values of f not bounded above on S?)

9. (5 pts each part, 25 pts total) Consider the integral

$$I = \int_{x=0}^{1} \int_{y=0}^{x} (x+y)\, dy dx.$$

a) Draw the region of integration.

b) Evaluate I.

c) Set up but **do not evaluate** I in the order $dxdy$.

d) Set up but **do not evaluate** I in polar coordinates, in the order $drd\theta$.

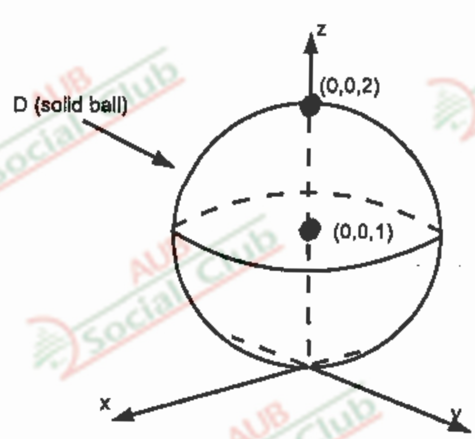
e) Set up but **do not evaluate** I in polar coordinates, in the order $d\theta dr$.

10. (7 pts each part, 28 pts total) In this problem, D is the solid ball with center $(0,0,1)$ and radius 1. The equations for the boundary of D in various coordinate systems are:

$$x^2 + y^2 + (z-1)^2 = 1, \qquad r^2 + (z-1)^2 = 1, \qquad \rho = 2\cos\varphi.$$

The density of D is $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$. We wish to compute the total mass of D.

The rest of problem 10 is on page iv.



Problem 10, continued.

a) Set up **but do not evaluate** an integral giving the total mass of D in **spherical** coordinates.

b) Evaluate the integral from part a).

c) Do the same as part a) (without evaluating!) in **cylindrical** coordinates.

d) Do the same as part a) (no evaluating!) in **rectangular** (i.e., xyz) coordinates.

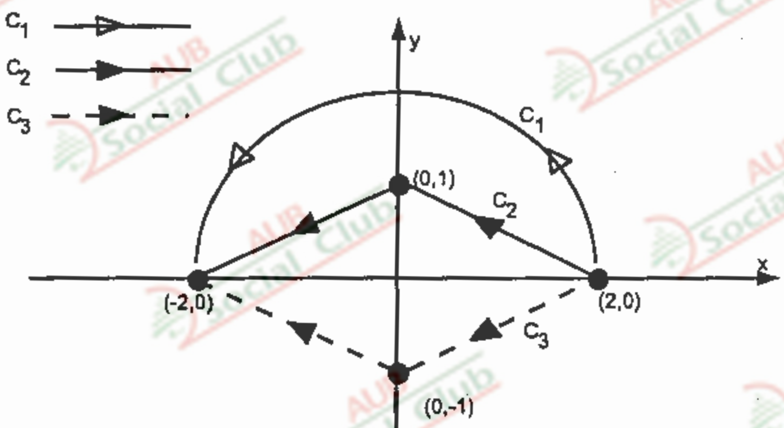
11. (7 pts each part, 14 pts total) Consider the two vector fields in the plane

$$\vec{F} = (0, x), \qquad \vec{G} = (0, y).$$

a) One of the two vector fields is conservative. Explain why, and find a potential function for the conservative field.

b) Let C be the cardioid, given in polar coordinates by $r = 1 + \cos\theta$. We orient C counterclockwise. Find the work integrals $\oint_C \vec{F} \cdot d\vec{r}$ and $\oint_C \vec{G} \cdot d\vec{r}$.

12. (6 pts each part, 18 pts total) The curves C_1, C_2, and C_3 are shown in the figure below. They all go from $(2, 0)$ to $(-2, 0)$.

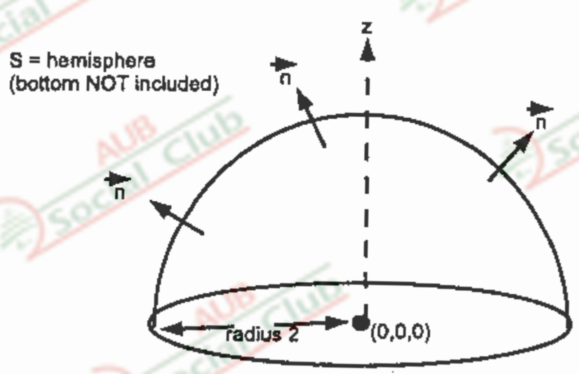


a) Compute the work integral $\int_{C_1} \vec{F} \cdot d\vec{r}$, where $\vec{F} = \dfrac{(-y, x)}{x^2 + y^2}$.

b) Do the same for the integral along C_2. (Hint: what is the curl of $\vec{F}$?)

c) Do the same for C_3, and use this to show that $\vec{F}$ is **not** conservative.

13. (10 pts) Given the vector field in space $\vec{F} = (x, y, z)$. Let S be the upper hemisphere of center $(0, 0, 0)$ and radius 2, oriented with a normal vector that points away from the origin. Compute the flux $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$ of $\vec{F}$ across S.



1. (5 pts each part, 15 pts total) Determine whether each of the following series converges or diverges:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^2-1}}{\sqrt{n^5+1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \tan^{-1} n$ (c) $\displaystyle\sum_{n=1}^{\infty} (n+1)(0.5)^n$

a) $\sum_{n=1}^{\infty} a_n$ with $0 \le a_n = \frac{\sqrt{n^2-1}}{\sqrt{n^5+1}} \le \frac{\sqrt{n^2}}{\sqrt{n^5}} = \frac{1}{n^{1.5}} = b_n$.

But $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{1.5}}$ converges (p-series, $p = 1.5 > 1$)

so $\sum_{n=1}^{\infty} a_n$ converges by the comparison test.

b) Here, $a_n = \tan^{-1} n$ and $\lim_{n \to \infty} a_n = \lim_{n \to +\infty} \tan^{-1} n = \pi/2 \neq 0$

$\therefore \sum_{n=1}^{\infty} a_n$ diverges by the nth term test

c) Use the root or ratio test to get $\rho = 0.5 < 1$, so the series converges.

OR use $(n+1) \ll (1.01)^n$ for large n

(polynomial growth << exponential growth)

so for large n, we have

$0 \le a_n = (n+1)(0.5)^n \ll (1.01)^n (0.5)^n = (0.505)^n \overset{\text{define}}{=\!=} b_n$

but $\sum_{n=1}^{\infty} b_n$ converges (geometric series, $r = 0.505 < 1$)

so $\sum_{n=1}^{\infty} a_n$ converges by comparison.

2. (12 pts) Find the interval of convergence of the following series. Remember to test the endpoints.

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{\sqrt{n}\,3^n}$$

We can use the root or ratio test to study this series.

For example, using the ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-5|^{n+1}}{\sqrt{n+1}\;3^{n+1}} \cdot \frac{\sqrt{n}\;3^n}{|x-5|^n} = \frac{|x-5|}{3\left(\sqrt{1+\frac{1}{n}}\right)}$$

and we see that $\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-5|}{3\sqrt{1}} = \frac{|x-5|}{3}$.

The interior of the interval of convergence corresponds to $\rho < 1$

$\iff |x-5| < 3 \iff 5-3 < x < 5+3 \iff 2 < x < 8.$

The endpoints are $x=8$ and $x=2$.

when $x=8$: we get $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}\,3^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series, $p = 0.5 \leq 1$)

$x=2$: we get $\sum_{n=1}^{\infty} \frac{(-3)^n}{\sqrt{n}\,3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. This is an alternating series with $u_n = \frac{1}{\sqrt{n}}$; when n increases, we have $\sqrt{n}$ increases so $\frac{1}{\sqrt{n}}$ decreases. Moreover, $\lim_{n\to\infty} u_n = 0$.

Hence the series converges for $x=2$ by the alternating series test. (Remark: in this case $\sum u_n = \sum \frac{1}{\sqrt{n}}$ diverges, so the convergence is conditional).

Final answer: the series converges for $2 \leq x < 8$

i.e. on the interval $[2, 8)$

3. (5 pts each part, 10 pts total)

a) Find the first four nonzero terms of the Taylor series for e^x and $\cos 3x$ centered at $a = 0$.

b) Compute the limit: $\lim_{x\to 0} \frac{e^x - 1 - x}{1 - \cos 3x}$.

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$\cos u = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \frac{u^6}{6!} + \cdots$

so $\cos 3x = 1 - \frac{3^2 x^2}{2!} + \frac{3^4 x^4}{4!} - \frac{3^6 x^6}{6!} + \cdots$

(you can simplify the fractions if you wish; but why bother? $1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - \frac{81}{80}x^6 + \cdots$)

b) $\frac{e^x - 1 - x}{1 - \cos 3x} = \frac{1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots - 1 - x}{1 - 1 + \frac{3^2x^2}{2!} - \frac{3^4x^4}{4!} + \cdots}$

$= \frac{\frac{x^2}{2!} + O(x^3)}{\frac{3^2x^2}{2!} + O(x^4)} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{\frac{1}{2!} + O(x)}{\frac{3^2}{2!} + O(x^2)} \xrightarrow[x\to 0]{} \frac{\frac{1}{2!}}{\frac{9}{2!}} = \boxed{\frac{1}{9}}.$

So the limit is $\boxed{\frac{1}{9}}$.

4. (6 pts each part, 12 pts total)

a) Find the second-order Taylor polynomial of the function $f(x) = \sqrt{x}$, **centered at** $x = 100$.

b) Using your answer P_2 above, give an approximate value for $\sqrt{99}$ and estimate the error.

a) $f(x) = \sqrt{x} = x^{1/2}$ $f(100) = 100^{1/2} = 10$ $c_0 = f(a) = 10$

$f'(x) = \frac{1}{2}x^{-1/2}$ $f'(100) = \frac{1}{2}\cdot(100)^{-1/2} = \frac{1}{20}$ $c_1 = f'(a) = \frac{1}{20}$

$f''(x) = -\frac{1}{4}x^{-3/2}$ $f''(100) = -\frac{1}{4}(100)^{-3/2} = -\frac{1}{4000}$ $c_2 = \frac{f''(a)}{2!} = \frac{-1}{8000}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

$$\boxed{P_2(x) = 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2}.$$

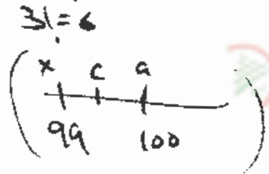
b) $\sqrt{99} = f(99) \approx P_2(99) = 10 + \frac{1}{20}(-1) - \frac{1}{8000}(-1)^2$

$= \boxed{10 - \frac{1}{20} - \frac{1}{8000}}$ is the approximation.

The error is $|R_2| = |f(99) - P_2(99)| = \left| f'''(c)\,\frac{(99-100)^3}{3!} \right|$

$= \left| \frac{3\ (-1)}{8\, c^{5/2}\ 3!} \right| = \frac{1}{16\, c^{5/2}}$ ($3! = 6$)

for some c between 99 & 100.



Note $99 \le c \le 100$

$\Rightarrow 99^{5/2} \le c^{5/2} \le 100^{5/2}$

$\Rightarrow \frac{1}{99^{5/2}} \ge \frac{1}{c^{5/2}} \ge \frac{1}{100^{5/2}}$

$\Longrightarrow \frac{1}{16\cdot 99^{5/2}} \ge \frac{1}{16\, c^{5/2}}$ & $\boxed{\text{the error is } \le \frac{1}{(16)(99^{5/2})}}$

To give you a feel for this, $99^{5/2}$ is about 2.5% less (why?)

$100^{5/2} = 10^5 = 100{,}000$, so $\frac{1}{16\,(99^{5/2})} \le \frac{1}{10\cdot(100)^{5/2}} = 10^{-6}$.

5. (7 pts each part, 14 pts total) We are given a function $f(x,y,z)$ [redacted] satisfying:

$$\vec{\nabla} f \Big|_{(1,2,3)} = (4,-5,6), \qquad \vec{\nabla} f \Big|_{(2,4,6)} = (8,-9,10), \qquad \vec{\nabla} f \Big|_{(2,0,1)} = (2,0,2).$$

a) Which of the following differences do you expect to be largest, and why?

$$f(1.1,2.1,3.1) - f(1,2,3), \qquad f(2.1,4.1,6.1) - f(2,4,6), \qquad f(2.1,0.1,1.1) - f(2,0,1).$$

b) Let $g(r,s) = f(r-s, r+s, 2rs)$. Find $\frac{\partial g}{\partial s}\Big|_{(r,s)=(3,1)}$.

a) we know $\vec{\nabla} f|_p \cdot \Delta\vec{r} \approx \Delta f$.

So for $f(1.1,2.1,3.1) - f(1,2,3) = \Delta f$, we have $\Delta \vec{r} = (0.1,0.1,0.1)$ and $p = (1,2,3)$, so $\vec{\nabla} f|_p = (4,-5,6)$, and

$$f(1.1,2.1,3.1) - f(1,2,3) \approx (4,-5,6)\cdot(0.1,0.1,0.1) = 0.4-0.5+0.6 = 0.5$$

similarly $f(2.1,4.1,6.1) - f(2,4,6) \approx (8,-9,10)\cdot(0.1,0.1,0.1) = 0.8-0.9+1 = 0.9$ (where $(8,-9,10) = \vec{\nabla} f|_{2,4,6}$)

$$f(2.1,0.1,1.1) - f(2,0,1) \approx (2,0,2)\cdot(0.1,0.1,0.1) = 0.4.$$

So (assuming the errors in these approximations are not too large*),

We expect that $f(2.1,4.1,6.1) - f(2,4,6) \approx 0.9$ is the largest of the three differences.

*(this depends on the sizes of f_{xx}, f_{xy}, and f_{yy} in the region)

b)
$$\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

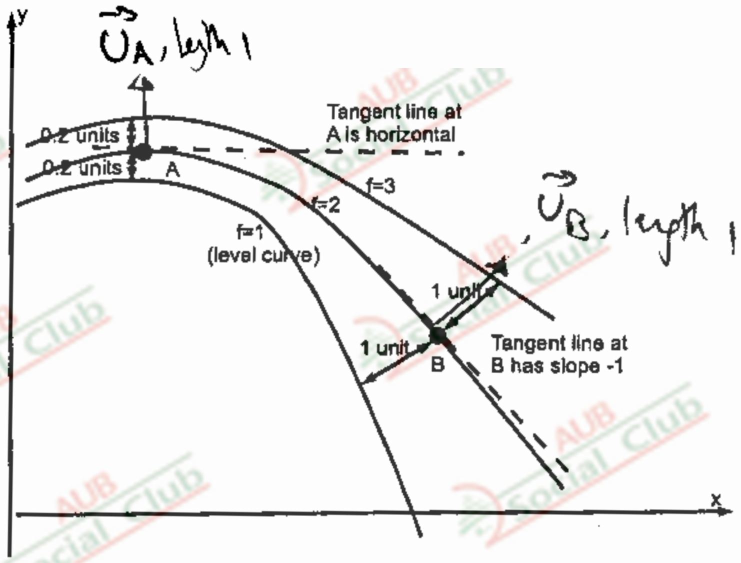
$$= \vec{\nabla} f\Big|_{(x,y,z) = (r-s,\, r+s,\, 2rs)} \cdot \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right).$$

At $(r,s) = (3,1)$ we have $(x,y,z) = (3-1, 3+1, 2\cdot3\cdot1) = (2,4,6)$ & $\vec{\nabla} f|_{(2,4,6)} = (8,-9,10)$. On the other hand

$$\left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right)\Big|_{(r,s)=(3,1)} = (-1,1,2r)\Big|_{(r,s)=(3,1)} = (-1,1,6)$$

so the answer is $(8,-9,10)\cdot(-1,1,6) = -8-9+60 = \boxed{43}$

6. (4 pts each part, 12 pts total) Some level curves of a function $f(x, y)$ are drawn below.



a) Give two **unit** vectors $\vec{u}_A, \vec{u}_B$ such that $\vec{u}_A$ points in the same direction as $\vec{\nabla} f \Big|_A$, and similarly $\vec{u}_B$ points in the same direction as $\vec{\nabla} f \Big|_B$.

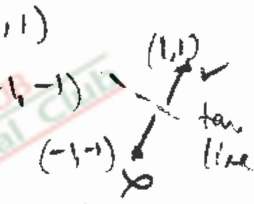
b) Based on the figure, what is a reasonable approximation to the directional derivatives

$$D_{\vec{u}_A} f \Big|_A, \qquad D_{\vec{u}_B} f \Big|_B?$$

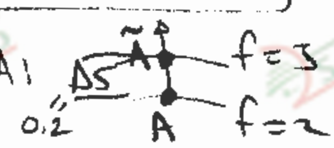
c) Deduce a reasonable approximation to each of the vectors $\vec{\nabla} f \Big|_A$ and $\vec{\nabla} f \Big|_B$.

a) see the picture: $\vec{U}_A$ must be $\perp$ the tangent line at A & pointing up (from f=2 to f=3, since f increases in the direction of $\vec{\nabla} f|_A$). Hence $\boxed{\vec{U}_A = (0,1)}$ (unit vector)

Similarly $\vec{U}_B$ is $\perp$ a line with slope -1, so $\vec{U}_B \uparrow\uparrow (1,1)$ and also pointing in that direction (NOT in the direction $(-1,-1)$ because of increasing f). $\therefore$ $\boxed{\vec{U}_B = \frac{1}{\sqrt{2}}(1,1) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) \text{ (unit vector)}}$.

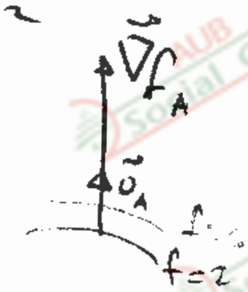


b) $D_{\vec{U}} f|_p \approx \frac{\Delta f}{\Delta s}$ in the direction of $\vec{U}$. at A:



$\Delta f = f(\tilde{A}) - f(A) = 3-2 = 1$

$\Delta s = 0.2$ so $\boxed{D_{\vec{U}_A} f|_A \approx \frac{1}{0.2} = 5}$



Similarly $\boxed{D_{\vec{U}_B} f|_B \approx \frac{\Delta f}{\Delta s} = \frac{1}{1} = 1}$

c) $D_{\vec{U}_A} f|_A = \vec{\nabla} f|_A \cdot \vec{U}_A = |\vec{\nabla} f|_A|$ since $\vec{\nabla} f|_A$ & $\vec{U}_A$ point in the same direction. $\therefore$ $\boxed{\vec{\nabla} f|_A = |\vec{\nabla} f|_A| \vec{U}_A \approx 5(0,1) = (0,5)}$

similarly $\boxed{\vec{\nabla} f|_B \approx 1(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$

7. (15 pts) Find and classify the critical points of the function $f(x,y) = \frac{x^4}{12} - 3xy + \frac{3y^2}{2}$.

$$\tilde{\nabla} f = (f_x, f_y) = \left(\frac{4x^3}{12} - 3y,\ -3x + \frac{6y}{2}\right) = \left(\frac{x^3}{3} - 3y,\ -3x + 3y\right)$$

Critical points occur when $\vec{\nabla} f = \vec{0}$ $\iff \begin{cases} x^3/3 - 3y = 0 \\ -3x + 3y = 0 \end{cases} \iff \begin{cases} y = \frac{x^3}{9} \\ y = x \end{cases}$

So we want $x = x^3/9$ or $\boxed{x^3 - 9x = 0} \iff x(x+3)(x-3) = 0$ (with $y = x$) $\iff x \in \{0, 3, -3\}$

we get three critical points $\boxed{P_1(0,0) \quad P_2(3,3) \quad P_3(-3,-3)}$.

Testing the critical points: the Hessian is $Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} x^2 & -3 \\ -3 & 3 \end{pmatrix}$

$Hf|_{P_1} = \begin{pmatrix} 0 & -3 \\ -3 & 3 \end{pmatrix}$, $\Delta = \det Hf = 0 - (-3)(-3) = -9 < 0$

$\therefore$ $\boxed{P_1(0,0) \text{ is a saddle point}}$.

$Hf|_{P_2} = \begin{pmatrix} 3^2 & -3 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ -3 & 3 \end{pmatrix}$, $\Delta = \det Hf = 27 - (+9) = 18 > 0$

$f_{xx}|_{P_2} = 9 > 0$

$\therefore$ $\boxed{P_2(3,3) \text{ is a local minimum}}$

$Hf|_{P_3} = \begin{pmatrix} (-3)^2 & -3 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ -3 & 3 \end{pmatrix}$, just like P_2

By the same reasoning, $\boxed{P_3(-3,-3) \text{ is a local minimum}}$.

(Challenge: show that P_2 & P_3 are global minima)
not part of the final

8. (15 pts total) Let $f(x,y,z) = 8x + y + z$, and consider the surface S given by $xyz = 1$ in the first octant only:

$$S = \{(x,y,z) \mid xyz = 1,\ x,y,z > 0\}.$$

a) (12 pts) Use Lagrange multipliers to find the minimum value of f on the surface S.

b) (3 pts) Show that f does not have a maximum value on S. (Why are the values of f not bounded above on S?)

a) $S : g(x,y,z) = 1$ where $g(x,y,z) = xyz$

to minimize f on S, we solve

$$\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g \\ g(x,y,z) = 1 \end{cases} \iff \begin{cases} (8,1,1) = \lambda(yz, xz, xy) \\ xyz = 1 \end{cases}$$

$$\iff \begin{cases} 8 = \lambda yz \\ 1 = \lambda xz \\ 1 = \lambda xy \\ xyz = 1 \end{cases} \Longrightarrow \begin{cases} 8x = \lambda \cdot xyz = \lambda \\ y = \lambda \cdot xyz = \lambda \\ z = \lambda \cdot xyz = \lambda \\ xyz = 1 \end{cases}$$

(multiply by x, multiply by y, multiply by z & use $xyz = 1$; $\Longleftarrow$ since $x,y,z \neq 0$ due to $xyz = 1$)

$$\iff \begin{cases} y = z = 8x \ (= \lambda) \\ xyz = 1 \end{cases} \iff \begin{cases} y = z = 8x \\ x(8x)(8x) = 1 \end{cases} \iff \begin{cases} y = z = 8x \\ 64x^3 = 1 \end{cases}$$

$$\iff \begin{cases} y = z = 8x \\ x = \frac{1}{\sqrt[3]{64}} = \frac{1}{4} \end{cases} \iff \boxed{(x,y,z) = \left(\tfrac{1}{4}, 2, 2\right)}$$

b) Here is one way: for any $t > 0$, we have $t, \frac{1}{t}, 1 > 0$ and $g(t, \frac{1}{t}, 1) = t \cdot \frac{1}{t} \cdot 1 = 1$, so $P_t(t, \frac{1}{t}, 1) \in S$

but $f(t, \frac{1}{t}, 1) = 8t + \frac{1}{t} + 1 > 8t$

& this can be made arbitrarily large by taking a large (positive!) t.

So f can attain arbitrarily large values at points of S

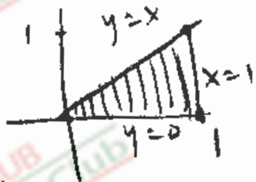
$\therefore$ no maximum value!

9. (5 pts each part, 25 pts total) Consider the integral

$$I = \int_{x=0}^{1} \int_{y=0}^{x} (x+y)\, dy dx.$$

a) Draw the region of integration.
b) Evaluate I.
c) Set up but **do not evaluate** I in the order $dxdy$.
d) Set up but **do not evaluate** I in polar coordinates, in the order $drd\theta$.
e) Set up but **do not evaluate** I in polar coordinates, in the order $d\theta dr$.

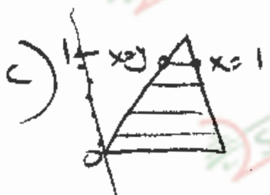
a)



b) $I = \int_{x=0}^{1} \left[xy + \frac{y^2}{2} \right]_{y=0}^{x} dx = \int_{x=0}^{1} \left[x \cdot x + \frac{x^2}{2} - 0 \right] dx = \int_{x=0}^{1} \frac{3x^2}{2} dx$

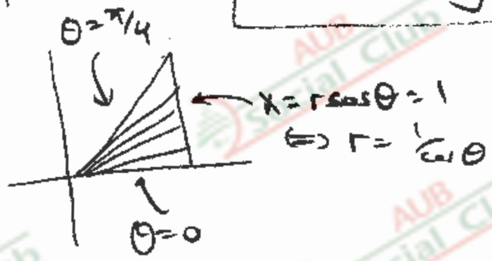
$= \left[\frac{x^3}{2} \right]_{x=0}^{1} = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}.$

c)



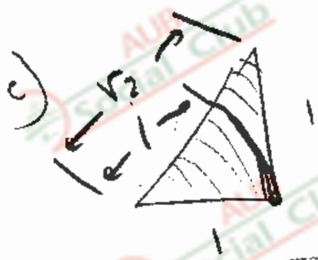
$$\int_{y=0}^{1} \int_{x=y}^{1} (x+y)\, dx\, dy.$$

d)

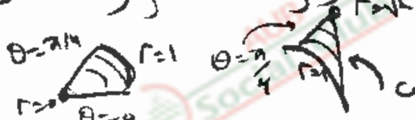


$$\int_{\theta=0}^{\pi/4} \int_{r=0}^{\frac{1}{\cos\theta}} \underbrace{(r\cos\theta + r\sin\theta)}_{x+y}\ \underbrace{r\,dr\,d\theta}_{dA}$$

e)



$I = \iint + \iint$



$\cos\theta = \frac{1}{r}$ & θ in 1st quadrant

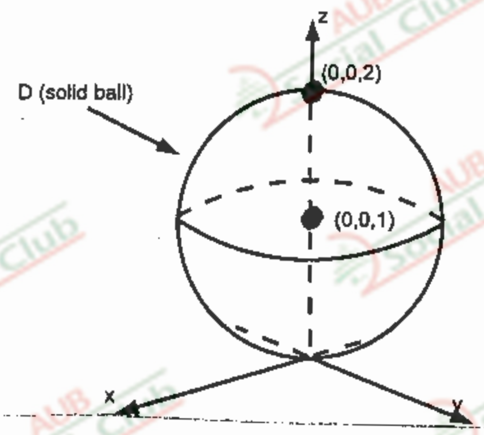
$\theta = \cos^{-1}(\frac{1}{r})$

So $$I = \int_{r=0}^{1} \int_{\theta=0}^{\pi/4} (r\cos\theta + r\sin\theta)\, r\, d\theta\, dr + \int_{r=1}^{\sqrt{2}} \int_{\theta=\cos^{-1}(\frac{1}{r})}^{\pi/4} (r\cos\theta + r\sin\theta)\, r\, d\theta\, dr.$$

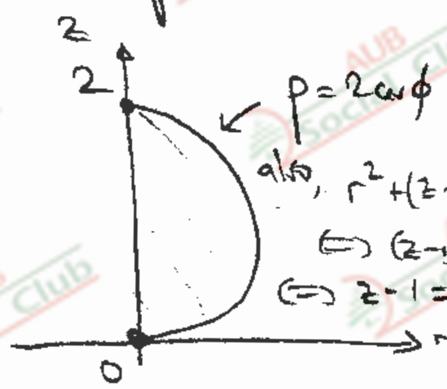
10. (7 pts each part, 28 pts total) In this problem, D is the solid ball with center $(0,0,1)$ and radius 1. The equations for the boundary of D in various coordinate systems are:

$$x^2+y^2+(z-1)^2=1, \qquad r^2+(z-1)^2=1, \qquad \rho=2\cos\varphi.$$

The density of D is $\delta(x,y,z)=\sqrt{x^2+y^2+z^2}$. We wish to compute the total mass of D.



rz plane: (fixed θ)



$\rho=2\cos\phi$

also, $r^2+(z-1)^2=1$

$\iff (z-1)^2=1-r^2$

$\iff z-1=\pm\sqrt{1-r^2}$

$\Updownarrow$

$z=1\pm\sqrt{1-r^2}$

Note $\delta=\sqrt{x^2+y^2+z^2}=\sqrt{r^2+z^2}=\rho$

a) Set up **but do not evaluate** an integral giving the total mass of D in **spherical** coordinates.

b) Evaluate the integral from part a).

c) Do the same as part a) (without evaluating!) in **cylindrical** coordinates.

d) Do the same as part a) (no evaluating!) in **rectangular** (i.e., xyz) coordinates.

a)  ϕ increasing from $\phi=0$ to $\phi=\pi/2$ ... 

$$\text{mass}=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{2\cos\phi}\delta\,dV=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{2\cos\phi}\rho\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

b) $$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\left[\frac{\rho^4}{4}\sin\phi\right]_{\rho=0}^{2\cos\phi}d\phi\,d\theta=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\left[\frac{16\cos^4\phi\sin\phi}{4}-0\right]d\phi\,d\theta$$

$$=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}4\cos^4\phi\sin\phi\,d\phi\,d\theta=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}-4\cos^4\phi\,d(\cos\phi)\,d\theta$$

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11. (7 pts each part, 14 pts total) Consider the two vector fields in the plane

$$\vec{F} = (0, x), \qquad \vec{G} = (0, y).$$

a) One of the two vector fields is conservative. Explain why, and find a potential function for the conservative field.

b) Let C be the cardioid, given in polar coordinates by $r = 1 + \cos\theta$. We orient C counterclockwise. Find the work integrals $\oint_C \vec{F} \cdot d\vec{r}$ and $\oint_C \vec{G} \cdot d\vec{r}$.

a) $\vec{F} = (M, N)$ $M = 0$, $N = x$ $\boxed{N_x = 1 \neq M_y = 0}$ so $\vec{F}$ cannot be conservative.

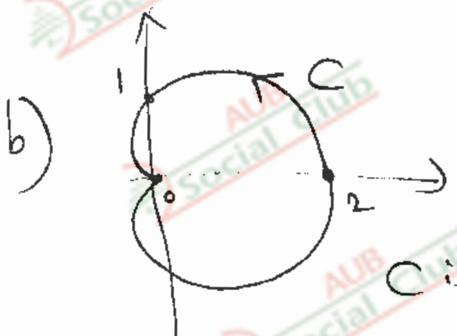
$\vec{G} = (\tilde{M}, \tilde{N})$ $\tilde{M} = 0$, $\tilde{N} = y$ $\tilde{N}_x = 0 = \tilde{M}_y$ so there's hope.

want $\begin{cases} g_x = 0 \Rightarrow g = g(y) \text{ only} \\ g_y = y \Rightarrow g = \frac{1}{2}y^2 \ (+c) \end{cases}$

we only need ONE potential: $\boxed{g = \frac{y^2}{2}}$.

& $\vec{G} = \vec{\nabla} g$

b)



C is a closed loop.

$\vec{G}$ is conservative so $\boxed{\oint_C \vec{G} \cdot d\vec{r} = 0}$.

for $\vec{F}$, you COULD parametrize using $\theta = t$, $x = r\cos\theta = (1+\cos\theta)\cos\theta = (1+\cos t)\cos t$

$y = r\sin\theta = (1+\cos\theta)\sin\theta = (1+\cos t)\sin t$

BUT this is messy.

It's much easier to use Green's: $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (N_x - M_y)\, dA$

$$= \iint_R 1\, dA \underset{\text{in polar}}{=} \int_{\theta=0}^{2\pi} \int_{r=0}^{1+\cos\theta} 1\, r\, dr\, d\theta$$

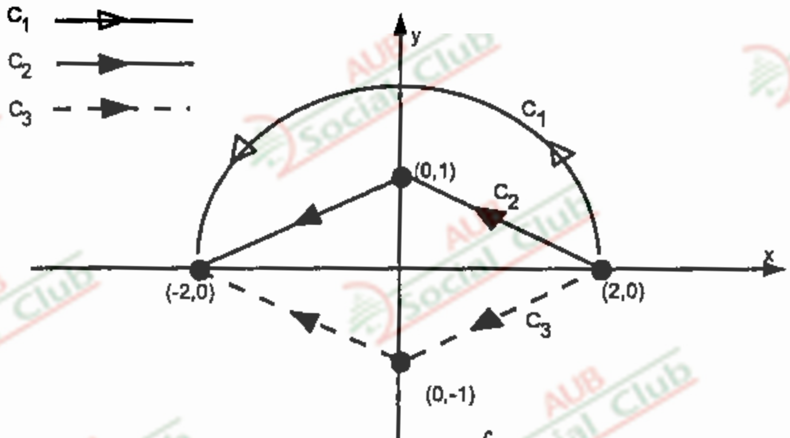
$$= \int_{\theta=0}^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{1+\cos\theta} d\theta = \int_{\theta=0}^{2\pi} \left[\frac{1 + 2\cos\theta + \cos^2\theta}{2} - 0\right] d\theta$$

← use $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ + continue

$$= \int_{\theta=0}^{2\pi} \left[\frac{3}{4} + \cos\theta + \frac{\cos 2\theta}{4}\right] d\theta = \left[\frac{3}{4}\theta + \sin\theta + \frac{\sin 2\theta}{8}\right]_{\theta=0}^{2\pi}$$

$$= \frac{3\pi}{2} + 0 + 0 - 0 = \boxed{\frac{3\pi}{2}}.$$

12. (6 pts each part, 18 pts total) The curves C_1, C_2, and C_3 are shown in the figure below. They all go from $(2,0)$ to $(-2,0)$.



a) Compute the work integral $\int_{C_1} \vec{F}\cdot d\vec{r}$, where $\vec{F} = \frac{(-y,x)}{x^2+y^2}$.

b) Do the same for the integral along C_2. (Hint: what is the curl of $\vec{F}$?)

c) Do the same for C_3, and use this to show that $\vec{F}$ is not conservative.

a) parametrize C_1: $\vec{r} = (2\cos t, 2\sin t)$ for $0 \le t \le \pi$, $d\vec{r} = \vec{v}\,dt = (-2\sin t, 2\cos t)\,dt$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{t=0}^{\pi} \vec{F}\Big|_{(2\cos t,\, 2\sin t)} \cdot (-2\sin t, 2\cos t)\,dt$$

$$= \int_{t=0}^{\pi} \frac{(-(2\sin t),\ 2\cos t)}{\underbrace{4\cos^2 t + 4\sin^2 t}_{4}} \cdot (-2\sin t, 2\cos t)\,dt = \int_{t=0}^{\pi} \frac{4\sin^2 t + 4\cos^2 t}{4}\,dt$$

$$= \int_{t=0}^{\pi} 1\,dt = \boxed{\pi}.$$

b) $+,+$

$$\operatorname{curl}\vec{F} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{(x^2+y^2) - 2x\cdot x}{(x^2+y^2)^2} - \frac{x^2+y^2 - 2y\cdot y}{(x^2+y^2)^2}$$

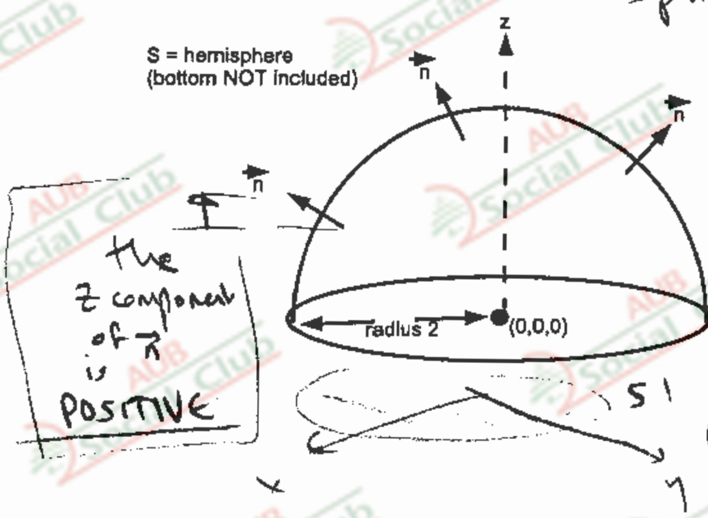
$= 0$. So by Green's theorem $\left(C_1, (-C_2) \right) = C_1 + (-C_2)$ note orientation

we have $\int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} = \iint (\operatorname{curl}\vec{F})\,dA = \iint 0\,dA = 0$

so $\int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} = \boxed{\pi}$. Note we were allowed to use Green's theorem because the singular ("bad") point of $\vec{F}$ at $(0,0)$ is <u>OUTSIDE</u> the region R: R, $(0,0)$

CONTINUED ON PAGE 15

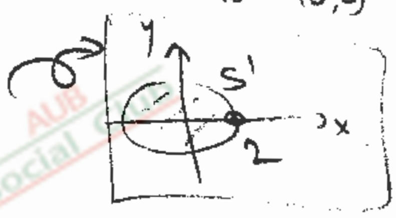
13. (10 pts) Given the vector field in space $\vec{F} = (x, y, z)$. Let S be the upper hemisphere of center $(0,0,0)$ and radius 2, oriented with a normal vector that points away from the origin. Compute the flux $\iint_S \vec{F} \cdot \vec{n}\, d\sigma$ of $\vec{F}$ across S.



sphere: $x^2+y^2+z^2=4$
but $z \geq 0$ for the hemisphere

$S: z = +\sqrt{4-x^2-y^2}$

the "shadow" S' of S is the disk of radius 2 & center $(0,0)$ in the xy plane



$\vec{n}\, d\sigma = \pm(-f_x, -f_y, 1)\, dx\, dy = +(-f_x, -f_y, 1)\, dx\, dy$

(since the z-component of $\vec{n}$ is positive) $= +\left(\frac{+2x}{2\sqrt{4-x^2-y^2}}, \frac{+2y}{2\sqrt{4-x^2-y^2}}, 1\right) dx\, dy$

note $\vec{F}\Big|_{z=\sqrt{4-x^2-y^2}} = \left(x, y, \sqrt{4-x^2-y^2}\right)$.

$$\therefore \iint_{(x,y,z)\in S} \vec{F}\cdot\vec{n}\, d\sigma = \iint_{(x,y)\in S'} \vec{F}\Big|_{z=f(x,y)} \cdot (-f_x, -f_y, 1)\, dx\, dy$$

$$= \iint_{(x,y)\in S'} \left(x, y, \sqrt{4-x^2-y^2}\right) \cdot \left(\frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1\right) dx\, dy$$

$$= \iint_{(x,y)\in S'} \left(\frac{x^2}{\sqrt{4-x^2-y^2}} + \frac{y^2}{\sqrt{4-x^2-y^2}} + \sqrt{4-x^2-y^2}\right) dA$$

rewrite as $\frac{4-x^2-y^2}{\sqrt{4-x^2-y^2}}$

$$= \iint_{(x,y)\in S'} \frac{x^2+y^2+4-x^2-y^2}{\sqrt{4-x^2-y^2}}\, dA \underset{\text{POLAR}}{=} \int_{\theta=0}^{2\pi}\int_{r=0}^{2} \frac{4}{\sqrt{4-r^2}}\, r\, dr\, d\theta$$

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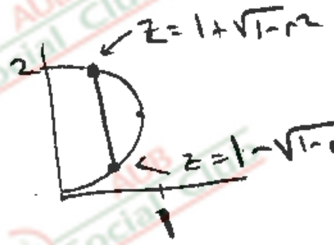
ex 10 continued

(10b) $= \int_{\theta=0}^{2\pi} \left[-\frac{4}{5}\cos^5\phi \right]_{\phi=0}^{\pi/2} d\theta = \int_{\theta=0}^{2\pi} \left(-\frac{4}{5}\cdot 0 + \frac{4}{5}\cdot 1 \right) d\theta$

$= \int_{\theta=0}^{2\pi} \frac{4}{5}\, d\theta = \boxed{\frac{8\pi}{5}}$.

c)



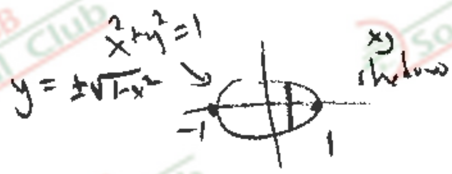
we get

$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=1-\sqrt{1-r^2}}^{1+\sqrt{1-r^2}} \underbrace{\sqrt{r^2+z^2}}_{\delta}\ \underbrace{r\,dz\,dr\,d\theta}_{dV}$$

(OR: $\int_{\theta=0}^{2\pi} \int_{z=0}^{2} \int_{r=0}^{\sqrt{1-(z-1)^2}} \sqrt{r^2+z^2}\cdot r\,dr\,dz\,d\theta$)

d)

← z slice: $1-\sqrt{1-x^2-y^2} \le z \le 1+\sqrt{1-x^2-y^2}$



xy shadow: disk of radius 1
center (0,0)

the easiest way is $\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{z=1-\sqrt{1-x^2-y^2}}^{1+\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$

(exercise: $\iiint_{z\ x\ y}$? Do via a z-shadow & an xy-slice

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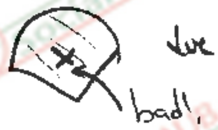
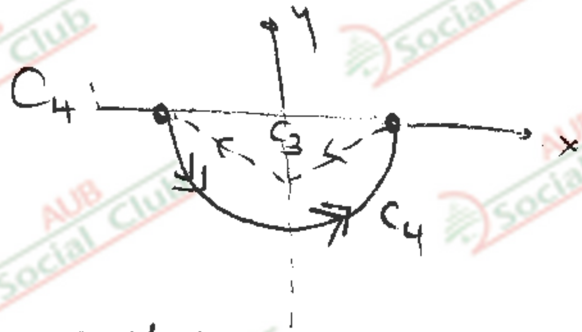
ex. 12, continued

2c) here the problen is that we cannot use Green's theorem on the region  due to the bad point (0,0).

One way: introduce a curve



C_4 ⟶⟶

C_3 - - → -

C_4: $\vec{r} = (2\cos t, 2\sin t)$ for $-\pi \leq t \leq 0$

Like in part a), $\displaystyle\int_{C_4} \vec{F} \cdot d\vec{r} = \int_{t=-\pi}^{0} 1\, dt = \pi.$

Now apply Green's theorem to the region R': 

whose oriented boundary is  $C_4 + C_3$

to get $\displaystyle\int_{C_4} \vec{F} \cdot d\vec{r} + \int_{C_3} \vec{F} \cdot d\vec{r} = \iint_{R'} 0\, dA = 0$

$\therefore \displaystyle\int_{C_3} \vec{F} \cdot d\vec{r} = -\int_{C_4} \vec{F} \cdot d\vec{r} = \boxed{-\pi}.$

Conclusion $\vec{F}$ is not conservative because

C_2 & C_3 have the same start & end points $((2,0) \to (-2,0))$

BUT $\displaystyle\int_{C_2} \vec{F} \cdot d\vec{r} = \pi \neq -\pi = \int_{C_3} \vec{F} \cdot d\vec{r}.$

(Remark $\vec{F}$ is the field "$\vec{\nabla}\theta$" that was discussed in class. The point is that the fake "potential" $\theta = \tan^{-1}(y/x)$ is NOT a uniquely defined function on the entire plane — when you go around the origin & come back to your starting point, θ has CHANGED by 2π.)

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Ex. 13, continued

so for: $\int_{\theta=0}^{2\pi}\int_{r=0}^{2}\frac{4r}{\sqrt{4-r^2}}\,dr\,d\theta$

$u = 4-r^2$

$$\left(\int\frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = \frac{u^{1/2}}{1/2} = 2u^{1/2}\ (+C)\right)$$

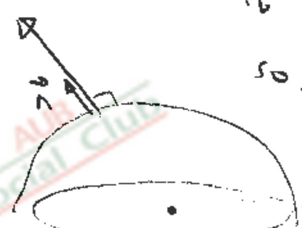
$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\frac{-2\,d(4-r^2)}{\sqrt{4-r^2}}\,d\theta$$

$$= \int_{\theta=0}^{2\pi}-2\left[2\sqrt{4-r^2}\right]_{r=0}^{2}d\theta = \int_{\theta=0}^{2\pi}-2\left(2\cdot 0-2\cdot 2\right)d\theta$$

$$= \int_{\theta=0}^{2\pi}+8\,d\theta = \boxed{\boxed{16\pi}}\,.$$

$\vec{F} = (x,y,z)$ so $\vec{F}\big|_P = \vec{OP}$ is radial

so $\vec{F} \uparrow\uparrow \vec{n}$



Note you can also do this geometrically:

so $\vec{F}\cdot\vec{n} = |\vec{F}| = \sqrt{x^2+y^2+z^2} = 2$ on S

& $\iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iint_S 2\,d\sigma = 2\cdot$(surface area of S, hemisphere of radius 2)

$$= 2\cdot\left(\frac{1}{2}\cdot 4\pi(2^2)\right) = \boxed{16\pi} \text{ again.}$$

Further note you can also use the divergence (Gauss') theorem on $S+T$ where T is the bottom flat surface oriented as shown: ($\vec{n}$ on S)

Details are left to you. This way, $S+T$ is the oriented boundary of the solid half-ball, and you have to compute $\iint_T \vec{F}\cdot\vec{n}\,d\sigma$ along the way. Luckily $\iint_T \vec{F}\cdot\vec{n}\,d\sigma = 0$ (check!)

FINAL EXAM.; MATH 201

February 6, 1998; 8:00-10:00 A.M.

Name: Signature:

Student number:

Section number (Encircle): 3 10 11 12

Instructors (Encircle): Prof. H. Abu-Khuzam Prof. A. Lyzzaik

1. Instructions:

   • No calculators are allowed.

   • There are two types of questions: **PART I** consisting of four subjective questions, and **PART II** consisting of twelve multiple-choice questions of which each has exactly one correct answer.

   • GIVE DETAILED SOLUTIONS FOR THE PROBLEMS OF **PART I** IN THE PROVIDED SPACE AND CIRCLE THE APPROPIATE ANSWERS FOR THE PROBLEMS OF **PART II**.

2. Grading policy:

   • 10 points for each problem of **PART I**.

   • 5 points for each problem of **PART II**.

   • 0 point for no answer, wrong answer, or more than one answer of **PART II**.

GRADE OF PART I/40:

GRADE OF PART II/60:

TOTAL GRADE/100:

Part I(1). Find the absolute maximum and minimum values of the function $f(x, y) = x^3 + 3xy - y^3$ on the triangular region R with vertices $(1, 2)$, $(1, -2)$, and $(-1, -2)$.

Part I(2). Evaluate the integral

$$\int_0^1 \int_x^{\sqrt{x}} e^{x/y} dy dx.$$

Part I(3). Set up a triple integral (without evaluating it)in cylindrical coordinates for the volume of the solid bounded by the $xy-$plane, the cylinder $r = 1 + \sin\theta$, and the plane $x + y + z = 2$.

Part I(4). Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{n5^n}(x-5)^n.$$

State where the series converges absolutely and conditionally.

Part II

1. The area of the region lying outside the circle $r = 3$ and inside the cardioid $r = 2(1 + cos\theta)$ is

   (a) $\frac{9}{2}\sqrt{3} - \pi$.

   (b) $\frac{9}{2}\sqrt{3} + \pi$.

   (c) $9\sqrt{3} + \pi/2$.

   (d) $9\sqrt{3} - \pi/2$.

   (e) None of the above.

2. The slope of the tangent line to the curve $r = 8\cos 3\theta$ at the point of the graph corresponding to $\theta = \pi/4$ is

   (a) 2.

   (b) -2.

   (c) 0.

   (d) 1.

   (e) None of the above.

3. If $f(x,y) = \frac{x^3y^2}{x^4+y^8}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$, then

   (a) $\lim_{(x,y)\to(0,0)} f(x,y) = 1/2$.

   (b) f is discontinuous at (0,0).

   (c) $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

   (d) $\lim_{(x,y)\to(1,-1)} f(x,y) = 2$.

   (e) None of the above.

4. An estimate to four decimal places of the value of the integral

$$\int_0^{0.1} x^2 e^{-x^2}\,dx \quad \text{is}$$

(a) 10^{-4}.

(b) 2×10^{-4}.

(c) 5×10^{-4}.

(d) 3×10^{-4}.

(e) None of the above.

5. The Maclaurin series of the integral

$$\int_0^x \sqrt[3]{1+t^2}\,dt \quad \text{is}$$

(a) $\sum_{n=1}^{\infty} \frac{(\frac{1}{3})(\frac{1}{3}-1)\cdots(\frac{1}{3}-n+1)}{n!(2n+1)} x^{2n+1}$.

(b) $x + \sum_{n=1}^{\infty} \frac{(\frac{1}{3})(\frac{1}{3}-1)\cdots(\frac{1}{3}-n+1)}{n!(2n+1)} x^{2n+1}$.

(c) $x - \sum_{n=1}^{\infty} \frac{(\frac{1}{3})(\frac{1}{3}-1)\cdots(\frac{1}{3}-n+1)}{n!(2n+1)} x^{2n+1}$.

(d) $x + \sum_{n=1}^{\infty} \frac{(\frac{1}{3})(\frac{1}{3}-1)\cdots(\frac{1}{3}-n+1)}{(2n+1)} x^{2n+1}$.

(e) None of the above.

6. If $a_n = (\frac{7}{2})^n + \frac{e^n}{n!}$ and $b_n = n^2(e^{1/n^2} - 1)$, then

(a) the sequences $\{a_n\}$ and $\{b_n\}$ diverge.

(b) the sequences $\{a_n\}$ and $\{b_n\}$ converge.

(c) the sequence $\{a_n\}$ diverges and $\{b_n\}$ converges.

(d) the sequence $\{a_n\}$ converges and $\{b_n\}$ diverges.

(e) None of the above.

7. The sum of the series

$$\sum_{n=0}^{\infty}[(-1)^n \frac{(\pi/2)^{2n+1}}{(2n+1)!} + \frac{n}{3^{n-1}}] \quad \text{is}$$

(a) 15/4.

(b) 5/4

(c) 7/4

(d) 13/4

(e) None of the above.

8. The function defined by $f(x,y) = \cos(\frac{x^3-y^3}{x^2+y^2})$ for $(x,y) \neq (0,0)$, and $f(0,0) = 1$

(a) has no limit at $(0,0)$.

(b) has a limit at $(0,0)$ but is not continuous at $(0,0)$.

(c) is continuous at $(0,0)$.

(d) is unbounded.

(e) None of the above.

9. If $z = f(x,y)$ where $x = e^r cos\theta$ and $y = e^r sin\theta$, then

(a) $f_x^2 - f_y^2 = e^{-2r}(f_r^2 - f_\theta^2)$.

(b) $f_x^2 + f_y^2 = e^{-2r}(f_r^2 + f_\theta^2)$.

(c) $f_x^2 + f_y^2 = e^{2r}(f_r^2 - f_\theta^2)$.

(d) $f_x^2 + f_y^2 = e^{2r}(f_r^2 + f_\theta^2)$.

(e) None of the above.

10. An equation of the tangent plane to the ellipsoid $\frac{3}{4}x^2 + 3y^2 + z^2 = 12$ at the point $P(2, 1, \sqrt{6})$ is

(a) $3x - 6y + 2\sqrt{6}z = 12$.

(b) $3y - 6x + 2\sqrt{6}z = 3$.

(c) $3y + 6x + 2\sqrt{6}z = 27$.

(d) $3x + 6y + 2\sqrt{6}z = 24$.

(e) None of the above.

11. If the directional derivatives of $f(x, y)$ at the point $P(1, 2)$ in the direction of the vector $\mathbf{i} + \mathbf{j}$ is $2\sqrt{2}$ and in the direction of the vector $-2\mathbf{j}$ is -3, then f increases most rapidly at P in the direction of the vector

(a) $3\mathbf{i} - \mathbf{j}$.

(b) $3\mathbf{i} + \mathbf{j}$.

(c) $\mathbf{i} - 3\mathbf{j}$.

(d) $\mathbf{i} + 3\mathbf{j}$.

(e) None of the above.

12. The function $f(x, y) = \frac{1}{3}x^3 + \frac{4}{3}y^3 - x^2 - 3x - 4y - 3$ admits

(a) a local maximum value $4/3$.

(b) a local minimum value $-42/3$.

(c) a saddle point $(3, 1, f(3, 1))$.

(d) an absolute maximum value $f(1, 1)$.

(e) None of the above.

January 27 , 2000

FINAL EXAMINATION
MATHEMATICS 201
FALL 1999- 2000

VERSION II

NAME -------------------

ID# --------------------

Time : 2 Hours

Circle out your instructor's name and your section number

Prof. H.Abu-Khuzam... Section: 6 and 12

Prof. A. Lyzzaik..Section: 5 and 7

NOTE: Make sure that you have 6 pages and twenty questions

GRADE: /100

Circle the correct answer in each of the following problems (5 points for each correct answer, and 0 for no or more than one answer)

1. The interval of convergence of the following series $\sum_{n=1}^{\infty} \frac{2^n (x-3)^n}{n^2}$ is

(A) $\frac{5}{2} < x < \frac{7}{2}$ (B) $\frac{5}{2} \le x \le \frac{7}{2}$ (C) $1 \le x \le 5$

(D) $1 < x < 5$ (E) none of the above

2. The surface $\frac{x^2}{36} - \frac{y^2}{36} = \frac{z}{6}$ is a

(A) elliptic paraboloid (B) elliptic cone (C) circular paraboloid

(D) hyperbolic paraboloid (E) none of the above

3. The sum of the series $\sum_{n=1}^{\infty} n(1/2)^{n-1}$ is

(A) 6 (B) e^2 (C) 4 (D) 25/4 (E) none of the above

4. Which one of the following series converges:

(A) $\sum_{n=1}^{\infty} \left[\sqrt{n+1} - \sqrt{n}\right]$ (B) $\sum_{n=1}^{\infty} \frac{2^n (n!)^3}{(3n)!}$ (C) $\sum_{n=1}^{\infty} \sqrt[n]{n}$

(D) $\sum_{n=1}^{\infty} \cos n\pi$ (E) $\sum_{n=1}^{\infty} \frac{1}{n}$

5. An estimate of the magnitude of the error obtained by taking the first four terms of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(0.01)^n}{n^3}$ to approximate its value is

(A) 1.56×10^{-12} (B) 8×10^{-11} (C) 8×10^{-13}

(D) 1.56×10^{-10} (E) none of the above

6. The sequence whose n-th term is $a_n=(1-\frac{\pi}{n})^{2n}$

(A) Converges to $e^{-\pi}$ (B) Diverges (C) Converges to $e^{2\pi}$

(D) Converges to $e^{-2\pi}$ (E) none of the above

7. The series $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n\sqrt{\ln n}}$

(A) is not alternating (B) diverges (C) converges conditionally

(D) converges absolutely (E) none of the above

8. The spherical coordinate equation for the sphere $x^2+y^2+(z-4)^2=16$ is

(A) $\rho=4\cos\theta$ (B) $\rho=4\cos\phi$ (C) $\rho=8\cos\phi$

(D) $\rho=8\cos\theta$ (E) none of the above

9. The value of the double integral Evaluate $\int_0^2 \int_{x/2}^1 e^{-y^2}\, dy dx$ is

(A) 1 - 1/e (B) 1 + 1/e (C) 1 (D) e (E) none of the above

10. If f(x,y) = ln xy +ln yz +ln xz, then the derivative of f at P(1,1,1) in the direction where f increases most rapidly is

(A) $\sqrt{3}$ (B) $2\sqrt{3}$ (C) 0

(D) $3\sqrt{3}$ (E) none of the above

11. Using the Maclaurin series for $\frac{-1}{1+x^5}$, the Maclaurin series of $\frac{5x^4}{(1+x^5)^2}$ is

(A) $\sum_{n=1}^{\infty} 5nx^{5n-1}$ (B) $\sum_{n=1}^{\infty} (-1)^n 5nx^{5n-1}$ (C) $\sum_{n=1}^{\infty} (-1)^{n+1} 5nx^{5n-1}$

(D) $\sum_{n=1}^{\infty} -5nx^{5n-1}$ (E) none of the above

12. Consider the function f(x,y) = $\begin{cases} \frac{x^4 y}{x^6 + y^3} & for(x, y) \neq (0,0) \\ k & for(x, y) = (0,0) \end{cases}$. Then

(A) f is discontinuous at (0,0) for all values of k.

(B) f is continuous at (0,0) provided that k=0

(C) f is continuous at (0,0), provided that k=1

(D) f is continuous at (0,0) , provided that k=-1.

(E) f is continuous at (0,0) for all values of k.

13. The function $f(x, y) = -x^3 - y^3 - 5xy - 1$ has

(A) saddle point at (0,0) and local maximum at $(-\frac{5}{3}, -\frac{5}{3})$.

(B) saddle point at (0,0) and local minimum at $(-\frac{5}{3}, -\frac{5}{3})$.

(C) saddle point at $(-\frac{5}{3}, -\frac{5}{3})$ and local minimum at (0,0).

(D) saddle point at $(-\frac{5}{3}, -\frac{5}{3})$ and local maximum at (0,0).

(E) local maximum at $(-\frac{5}{3}, -\frac{5}{3})$.

14. The function $f(x, y) = 3xy - 6x - 3y + 7$ defined on the triangular plate R with vertices (0,0), (3,0), and (0,5) has

(A) an absolute maximum $\frac{9}{5}$ and absolute minimum –8.

(B) an absolute maximum $\frac{9}{5}$ and absolute minimum –11.

(C) an absolute maximum 7 and absolute minimum 1.

(D) an absolute maximum 8 and absolute minimum –11.

(E) none of the above.

15. The area of the region that is inside the cardioid $r = 4 + 4\cos\theta$ and outside the circle $r = 6$ is

(A) $18\sqrt{3} + 2\pi$ (B) $18\sqrt{3} - 4\pi$ (C) 5π

(D) $5\sqrt{3} + 10$ (E) none of the above

16. Using triple integration in cylindrical coordinates, the volume of the solid bounded above by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, below by the xy-plane, and laterally by the cylinder $x^2 + y^2 = 9$ is

(A) $40\pi/3$ (B) $35\pi/3$ (C) $122\pi/3$ (D) $46\pi/3$ (E) none of the above

(2)

American University of Beirut

MATH 201

Calculus and Analytic Geometry III

Fall 2004

Final Exam

Name: .... ID #: ....

Exercise 1 a) *[2 points]*: Find $\lim_{n\to+\infty} \frac{\ln^n n}{n}$.

(hint: this is the same sequence of quiz 1)

b) *[3 points]*: What can you say about the series $\sum_{n=1}^{+\infty} \frac{\ln^n n}{n^2}$? Justify.

Exercise 2 a) *[3 points each]* Discuss whether the following series converges or diverges.

i) $\sum_{n=1}^{+\infty} (-1)^n \frac{\sin(n\frac{\pi}{2})}{n^2}$

ii) $\sum_{n=0}^{+\infty} \frac{1}{2^{n^2}}$

iii) $\sum_{n=1}^{+\infty} n^2 \sin(1/n)\tan(1/n)$

b) *[8 points]* Find the interval of convergence of the power series $\sum_{n=1}^{+\infty} \frac{\ln n}{n}(x+2)^n$

(be sure to check at the end points)

Exercise 3 a) *[3 points]* Find the equation of the tangent plane to the surface $z = x^5 - x^2y^2 + 4$ at the point $(1, 1, 4)$.

b) *[5 points]* Suppose that the equation $x^5 - x^2y^2 + 2yz - 8 = 0$ defines x as a function of y and z. Find the values of $\frac{\partial x}{\partial y}$ and $\frac{\partial x}{\partial z}$ at the point $(1, 1, 4)$.

c) *[3 points]* Prove or disprove: $g(x,y) = \frac{x^6y^4}{x^2+y^2}$ can be extended by continuity at $(0, 0)$. Justify.

Exercise 4 *[15 points]* Find the absolute minimum and maximum values of the function $f(x, y) = x^2 + y^2 + 2x - y + 1$ on the domain R defined by $\{(x, y) \in \mathrm{R}^2; x^2 + y^2 \leq 4 \text{ and } y \geq 0\}$.

Exercise 5 *[10 points]* Evaluate $I = \int_0^1 \int_z^1 \int_0^x e^{x^3}\, dydxdz$.

Exercise 6 *[12 points]* Let V be the volume of the region R bounded laterally by the cylinder $x^2 + (y-1)^2 = 1$, from above by the cone $z = \sqrt{x^2+y^2}$, and from below by the xy-plane.

a) Sketch the region of integration.

b) Express V as iterated triple integral in cartesian coordinates in the order $dzdxdy$ *(do not evaluate the resulting integral).*

c) Express V as iterated triple integral in cylindrical coordinates, then evaluate the resulting integral.

Exercise 7 *[15 points]* Use the transformation $u = y/x, v = xy$ to find

$$\int\int_R e^{xy}\,dA$$

over the region R in the first quadrant enclosed by the lines $y = x, y = x/2$ and the hyperbola $y = 1/x$ and $y = 2/x$.
(sketch both regions of integration)

Exercise 8 *[15 points]*

a) Write the complete statement of Green's Theorem.

b) Find $\oint_C (x^2 - y)dx + xdy$, where C is the closed curve positively directed given by the equations $x^2 + y^2 = 4$, and $y \geq 0$.

   i) By evaluating directly the line integral.

   ii) By using Green's Theorem quoted above.

FINAL EXAM.; MATH 201

January 28, 2004

Name: Signature:

Student number:

Section number (Encircle):

17 (Miss Jaafar) 18 (Miss Jaafar) 19 (Mrs. Jurdak) 20 (Mr. Lyzzaik)

Instructor: Prof. Abdallah Lyzzaik

1. Instructions:

- Calculators are allowed.

- There are two types of questions: **PART I** consists of **six subjective** questions, and **PART II** consists of **seven multiple-choice** questions of which each has exactly one correct answer.

- GIVE DETAILED SOLUTIONS FOR THE PROBLEMS OF **PART I** IN THE PROVIDED SPACE AND CIRCLE THE APPROPRIATE ANSWER FOR EACH PROBLEM OF **PART II**.

2. Grading policy:

- 12 points for each problem of **PART I**.

- 4 points for each problem of **PART II**: 0 point for no answer, -1 for a wrong answer or more than one answer of **PART II**.

GRADE OF PART I/72:

GRADE OF PART II/28:

TOTAL GRADE/100:

Part I (1). Find the absolute maximum and minimum values attained by the function $f(x,y) = xy - x - y + 3$ on the triangular region R in the xy-plane with vertices $(0,0)$, $(2,0)$, and $(0,4)$.

Part I (2). Evaluate the integral

$$\int_0^2 \int_{y/2}^1 y\, e^{x^3}\, dx\, dy.$$

Part I (3). Set up a triple integral (without evaluating it) in cylindrical coordinates for the volume of the solid bounded by the xy-plane, the cylinder $r^2 = \cos 2\theta$, and the sphere $x^2 + y^2 + z^2 = 1$.

Part I (4). Evaluate the integral

$$\int\int_R \sin\left(\frac{y-x}{y+x}\right)\, dx\, dy,$$

where R is the trapezoid in the $xy-$plane with vertices $(1,1)$, $(2,2)$, $(4,0)$, and $(2,0)$, by making the change of variables: $u = y - x$, $v = y + x$.

Part I (5). Evaluate the line integral

$$\oint_C 3xy\,dx + 2x^2\,dy,$$

where C is the boundary of the region R bounded above by the line $y = x$ and below by the parabola $y = x^2 - 2x$. Interpret this integral in terms of vector fields.

Part I (6). Find the interval of convergence of the power series

$$\sum_{n=2}^{\infty} \frac{(2x-1)^n}{\ln n}.$$

State where the series converges absolutely and conditionally.

Part II

1. If $f(x, y) = 2x^2y/(x^4 + y^2)$, then

   (a) $\lim_{(x,y)\to(0,0)} f(x, y) = 0$.

   (b) $\lim_{(x,y)\to(0,0)} f(x, y) = 1$.

   (c) $\lim_{(x,y)\to(0,0)} f(x, y) = 2$.

   (d) $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

   (e) None of the above.

2. An estimate of the integral

$$\int_0^1 \frac{1 - \cos x}{x^2}\, dx$$

with an error less than $1/(6!5)$ is

   (a) $1/2! + 1/(4!3)$.

   (b) $1/2! - 1/(4!3) + 1/(6!5)$.

   (c) $-1/2! + 1/(4!3) - 1/(6!5)$.

   (d) $1/2! - 1/(4!3)$.

   (e) None of the above.

3. The function defined by

$$f(x, y) = \tan\left(\frac{x^3 - y^3}{x^2 + y^2}\right)$$

for $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$

   (a) is continuous at $(0, 0)$.

   (b) has no limit at $(0, 0)$.

   (c) has a limit at $(0, 0)$ but is discontinuous at $(0, 0)$.

   (d) is bounded in the $xy-$plane.

   (e) None of the above.

4. If $w = f(x, y)$ where $x = e^r \cos\theta$ and $y = e^r \sin\theta$, then

(a) $w_{xx} + w_{yy} = w_{rr} + w_r/r + w_{\theta\theta}/r^2$.

(b) $w_{xx} + w_{yy} = -w_{rr} + w_r/r + w_{\theta\theta}/r^2$.

(c) $w_{xx} + w_{yy} = w_{rr} + w_r/r - w_{\theta\theta}/r^2$.

(d) $w_{xx} + w_{yy} = w_{rr} - w_r/r + w_{\theta\theta}/r^2$.

(e) None of the above.

5. An equation of the tangent plane to the surface with equation $z^3 + xz - y^2 = 1$ at the point $(1, 3, 2)$ is

(a) $2x + 6y + 13z = 10$.

(b) $2x + 6y - 13z = 10$.

(c) $2x - 6y + 13z = -10$.

(d) $2x - 6y + 13z = 10$.

(e) None of the above.

6. The volume of the solid bounded by the cylinder $y = x^2$ and the planes $y + z = 4$ and $z = 0$ is given by the triple integral

(a) $\int_0^4 \int_0^{4-y} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$.

(b) $2 \int_0^4 \int_{\sqrt{y}}^{2} \int_0^{4-y} dz\, dx\, dy$.

(c) $\int_{-2}^{2} \int_0^{x^2} \int_0^{4-y} dz\, dy\, dx$.

(d) $\int_0^4 \int_0^{4-z} \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy\, dz$.

(e) None of the above.

7. The function $f(x,y) = x^3 + 3xy - y^3$ admits

(a) a saddle point and a local minimum value.

(b) no saddle point and no local minimum value.

(c) a local minimum value and no saddle point.

(d) a saddle point and no local minimum value.

(e) None of the above.

Final - Fall 2003-2004 (January 28) (3) (Makdisi) (C)

PART [illegible]ICK ANSWERS, NO JUSTIFICATION REQUIRED, NO PARTIAL CREDI

1 (2 pts/part, total 10 pts). Which of the following series converge and which diverge? Circle your answer.

1a) $\sum_{n=1}^{\infty} \frac{2^{n+2}}{3^{n-2}}$ Converges Diverges

1b) $\sum_{n=1}^{\infty} (-1)^n$ Converges Diverges

1c) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Converges Diverges

1d) $\sum_{n=1}^{\infty} \frac{\sin(n^2)}{n^2}$ Converges Diverges

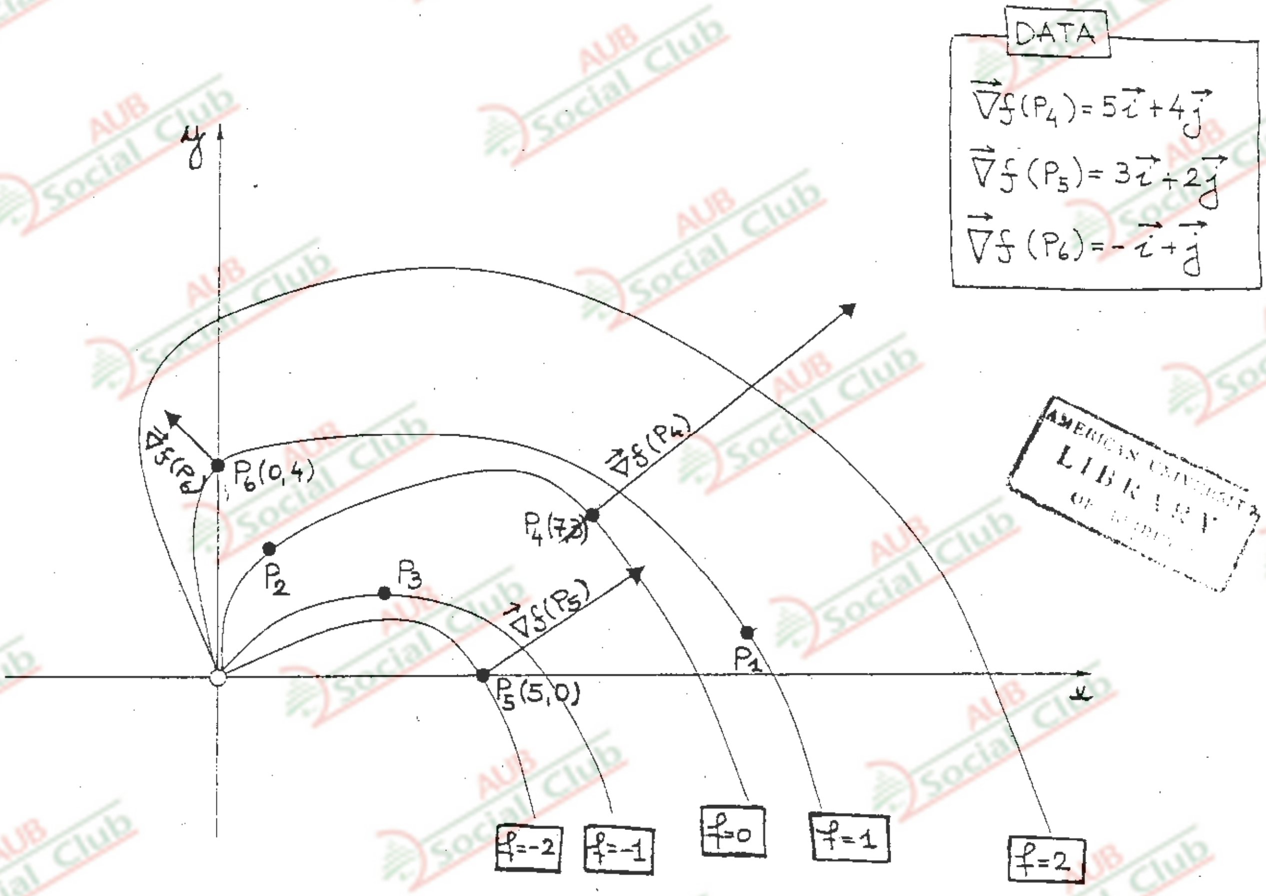
1e) $\sum_{n=1}^{\infty} \frac{(1+1/n)^n}{n}$ Converges Diverges

2 (2pts). Fill in the blank: the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{n!\,(n+1)!}{(2n)!} x^n$ is $R =$ ______________.

3 (1pt/part, total 4pts. Note 1 blank = 1 part.). Fill in the blanks for the first few coefficients in the Maclaurin series for the following integral:

$$\int_{t=0}^{x} \frac{\sin t}{t}\, dt = \_\_\_\_\_\_ + \_\_\_\_\_\_ x + \_\_\_\_\_\_ x^2 + \_\_\_\_\_\_ x^3 + \ldots$$

4 (2pts each for parts a–d, 4pts each for parts e–f, total 16pts). The following picture shows level curves for a function $f(x, y)$, and the value of the gradient $\vec{\nabla} f$ at some points.



4a) Draw a vector on the figure at P_1 pointing in the direction of maximum increase of f.

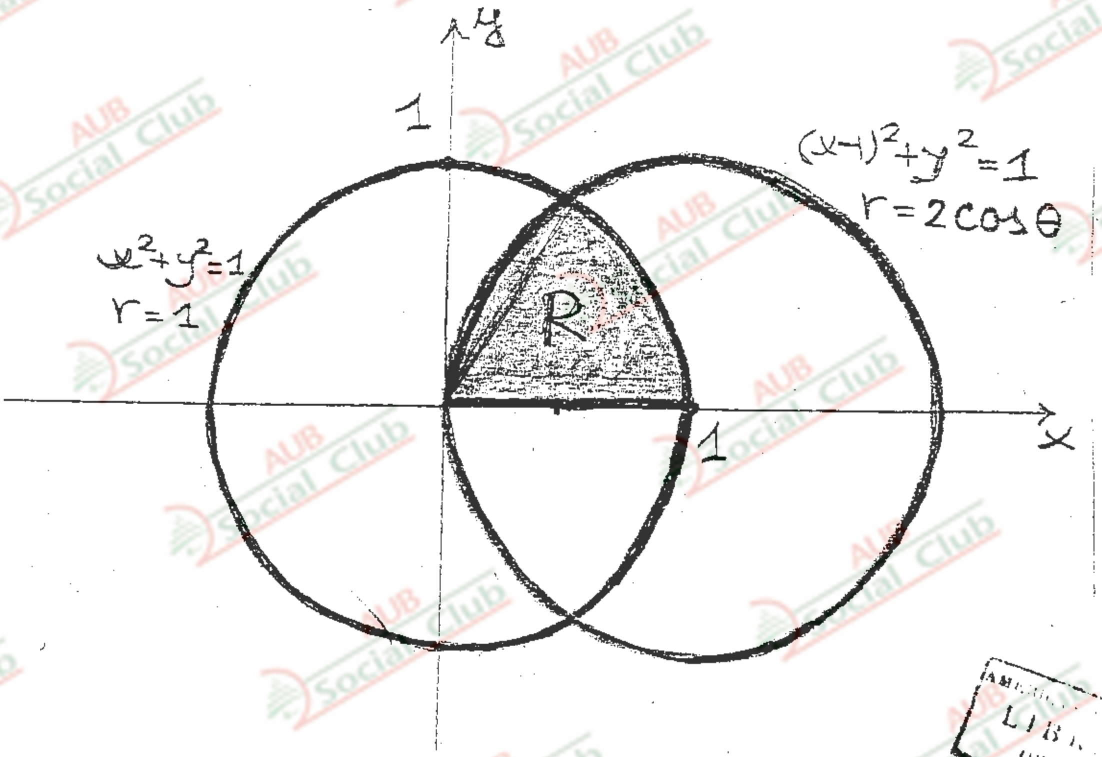
Circle the correct answer for 4b, 4c, 4d:

4b)	$\left.\frac{\partial f}{\partial x}\right\|_{P_2}$	is	positive	zero	negative	does not exist.
4c)	$\left.\frac{\partial f}{\partial x}\right\|_{P_3}$	is	positive	zero	negative	does not exist.
4d)	$\lim_{(x,y)\to(0,0)} f(x, y)$	is	positive	zero	negative	does not exist.

4e) Fill in the blank: The equation of the tangent line to the level curve passing through $P_4(7, 3)$ is ______________________.

4f) Fill in the blank: We start from the point $P_5(5, 0)$, and move a distance of $ds = \frac{1}{100}$ units in the direction of the vector $\bar{v} = \bar{i} + 2\bar{j}$. Then the change in the value of $f(x, y)$ is approximately ____________.

5 (0.5 pt/part, total 10.5pts. Note 1 blank = 1 part.). Consider the shaded region R below (R is common to the two circles shown and lies above the x-axis.) Fill in the blanks for the following integrals in rectangular and polar coordinates:



5a) $\displaystyle\iint_R 1\,dA = \int_{y=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{x=\_\_\_\_\_\_}^{\_\_\_\_\_\_} dx\,dy.$

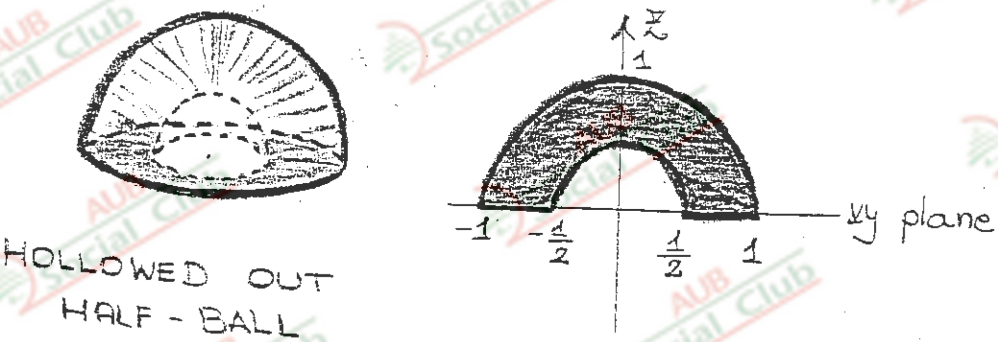
5b) $\displaystyle\iint_R 1\,dA = \int_{x=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{y=\_\_\_\_\_\_}^{\_\_\_\_\_\_} dy\,dx$

$\displaystyle + \int_{x=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{y=\_\_\_\_\_\_}^{\_\_\_\_\_\_} dy\,dx.$

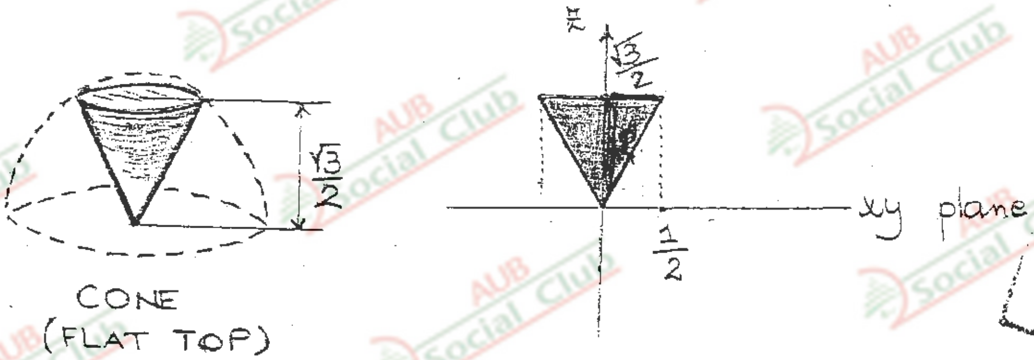
5c) $\displaystyle\iint_R 1\,dA = \int_{\theta=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{r=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \_\_\_\_\_\_\ dr\,d\theta$

$\displaystyle + \int_{\theta=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{r=\_\_\_\_\_\_}^{\_\_\_\_\_\_} [same]\,dr\,d\theta.$

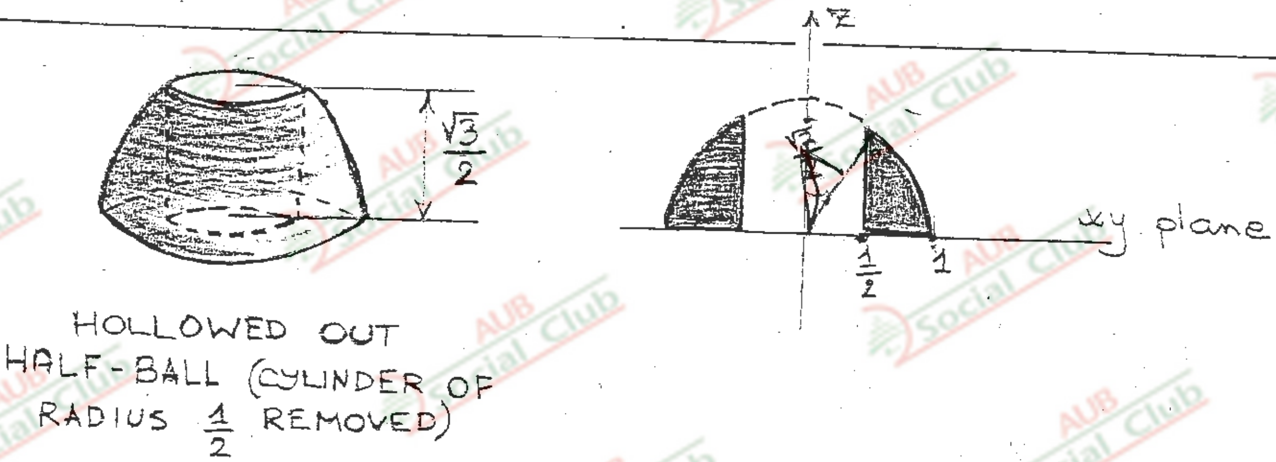
6 (0.5 pts/part, total 9.5 pts. Note 1 blank = 1 part.). In each of the pictures below, we give a 3-dimensional picture and a cross-section of a region D (which is always part of the half-ball $x^2+y^2+z^2 \leq 1, z \geq 0$). In each case, fill in the blanks for integration in spherical coordinates:



6a) $\iiint\limits_D 1\, dV = \int_{\theta=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\rho=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \_\_\_\_\_\_ \, d\rho\, d\varphi\, d\theta.$

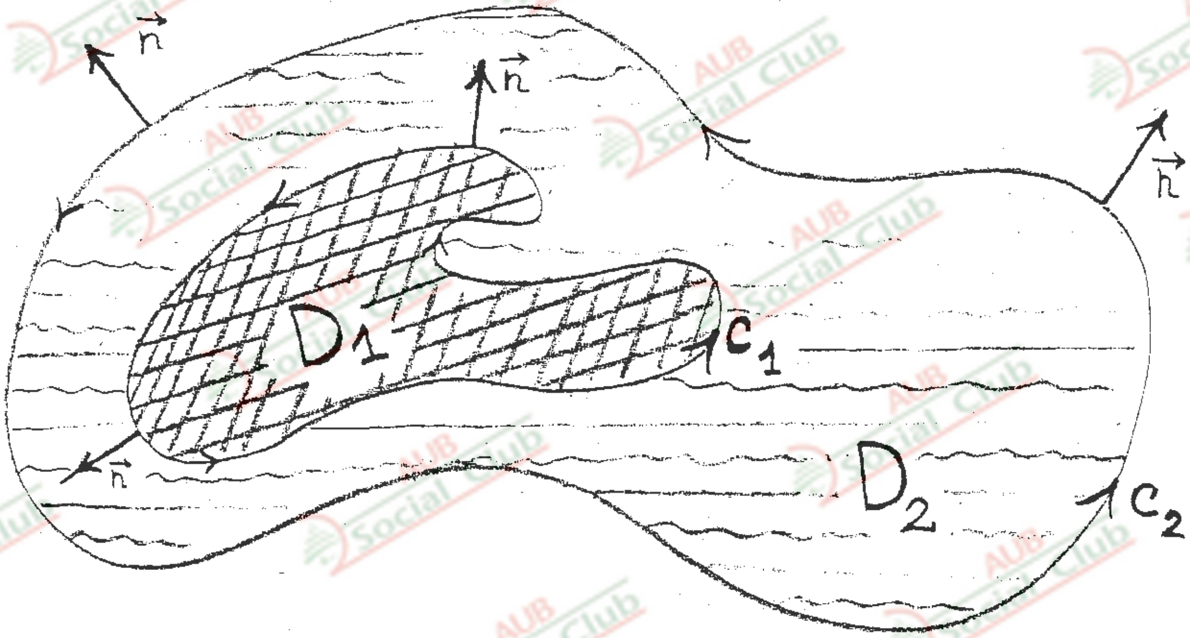


6b) $\iiint\limits_D 1\, dV = \int_{\theta=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\rho=\_\_\_\_\_\_}^{\_\_\_\_\_\_} [same]\, d\rho\, d\varphi\, d\theta.$



6c) $\iiint\limits_D 1\, dV = \int_{\theta=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_\_}^{\_\_\_\_\_\_} \int_{\rho=\_\_\_\_\_\_}^{\_\_\_\_\_\_} [same]\, d\rho\, d\varphi\, d\theta.$

7 (2 pts/part, total 8 pts. Note 1 blank = 1 part.). The following picture shows two curves C_1 and C_2 in the plane. D_1 is the region inside C_1. D_2 is the region that is inside C_2 AND outside C_1. We go around each of C_1 and C_2 counterclockwise, and the normal vector points outwards of C_1 and C_2 in each case, as drawn on the figure.



State Green's theorem for $\vec{F} = x\vec{i} + xy\vec{j} = (x, xy)$:

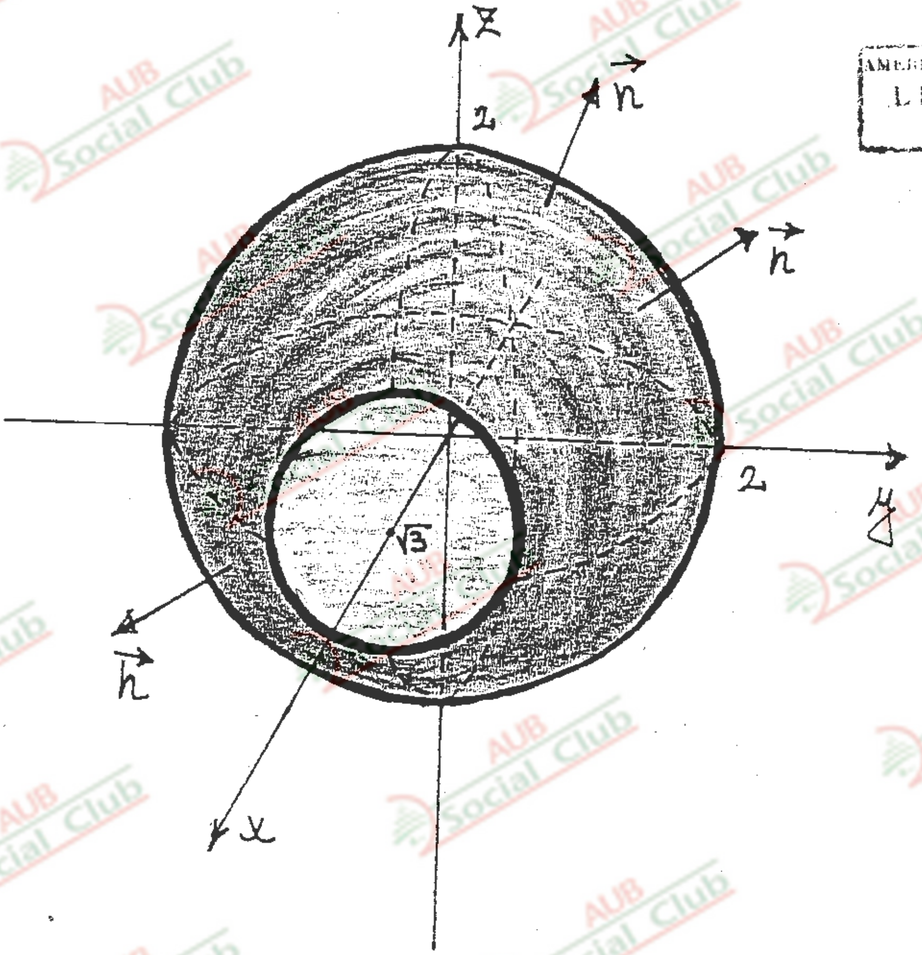
7a) $\iint_{D_1}$ ____________ $dA = \int_{C_1} \vec{F} \cdot \vec{T}\, ds.$

7b) $\iint_{D_2}$ ____________ $dA = \frac{}{(+ \text{ or } -?)} \int_{C_1} \vec{F} \cdot \vec{n}\, ds + \frac{}{(+ \text{ or } -?)} \int_{C_2} \vec{F} \cdot \vec{n}\, ds.$

8 (1pt/part, 4 pts total). Recall that Stokes' Theorem says:

$$\iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n} \, d\sigma = \int_C \vec{F} \cdot d\vec{r},$$

where the curve C is the boundary of the surface S, and we go around C in the direction compatible with $\vec{n}$, according to the right hand rule. Below is a picture where S is the part of the sphere of radius 2 ($x^2 + y^2 + z^2 = 4$) with $x \leq \sqrt{3}$; the normal vector $\vec{n}$ points away from the origin. C is the circle which is the boundary of S.



8a) On the figure, indicate the direction that one has to go around C for Stokes' Theorem.

8b) Parametrize C in the orientation that you have described. You need to enter a constant number in each of the blanks below:

$(x(t), y(t), z(t)) = (\underline{\ \sqrt{3}\ }, \underline{\qquad} \cos t, \underline{\qquad} \sin t), \qquad 0 \leq t \leq 2\pi.$

PART II. FULL SOLUTIONS REQUIRED, PARTIAL CREDIT AVAILABLE.

9 (10 pts). 9a) Find the second degree Taylor approximation, centered at $a = 2$, to the function $f(x) = \ln x$. (Your answer should have the form $P_2(x) = c_0 + c_1(x-2) + c_2(x-2)^2$ for appropriate c_0, c_1, c_2.)

9b) Use the Remainder Theorem for Taylor series to show that:

$$\text{if } 1.9 \leq x \leq 2.1, \text{ then } \left|\ln(x) - P_2(x)\right| \leq \frac{1}{18000}.$$

[Useful information so you can avoid doing long calculations by hand :
$3 < (1.9)^2 < 4$, $\quad 6 < (1.9)^3 < 7$, $\quad 4 < (2.1)^2 < 5$, $\quad 9 < (2.1)^3 < 10$.]

10 (10 pts). Evaluate the following limit:

$$\lim_{n\to\infty} \frac{(1+\frac{1}{n})^{5/2} \quad -1-\frac{5}{2n}}{e^{1/n^2}-1}.$$

11 (10 pts). We are given a function $f(x,y)$ such that:

$$\vec{\nabla} f \Big|_{(7,3)} = 5\vec{i} + 4\vec{j}, \qquad \vec{\nabla} f \Big|_{(5,0)} = 3\vec{i} + 2\vec{j}, \qquad \vec{\nabla} f \Big|_{(0,4)} = -\vec{i} + \vec{j}.$$

Assume that $x = x(s,t) = s^2 + t^2$ and $y = y(s,t) = st^2 - 4$. Find the value of

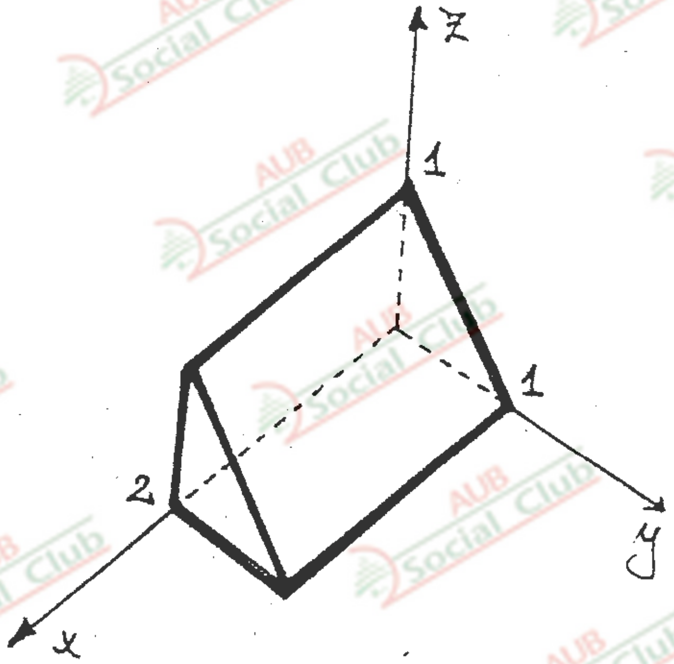
$$\frac{\partial f(x(s,t), y(s,t))}{\partial s} \Big|_{(s,t)=(1,2)}$$

12 (15 pts). We are given the function $f(x, y, z) = z^2 - 2z + xy$, and we **restrict** (x, y, z) **to lie in the solid cylinder** $D : x^2 + y^2 \leq 2$ (and z arbitrary). At what point(s) of D does f attain its **minimum** value?

(Hints: (i) remember to check the interior and the boundary of D, (ii) remember that we are in three dimensions, (iii) do **not** try to do the second derivative test — we didn't cover it in class for three dimensions!)

13 (8 pts). Let R be the region in the **first quadrant** underneath the parabola $y = 3-3x^2$. Find the average value of $f(x, y) = 2x$ on the region R.

1$\frac{1}{2}$ (8 pts).



Let D be the region in the **first octant** cut out by the planes $y + z = 1$ and $x = 2$. (See the figure.) The density of D is given by $\delta(x, y, z) = x^2$. Find the total mass of D.

15 (12 pts). Given the two vector fields in the plane:

$$\vec{F} = x\vec{j} = (0, x), \qquad \vec{G} = (2xy + 1)\vec{i} + (x^2 + e^y)\vec{j} = (2xy + 1, x^2 + e^y).$$

15a) For each of $\vec{F}$ and $\vec{G}$, **either** show that the vector field is not conservative, or show that the field is conservative by finding a potential function.

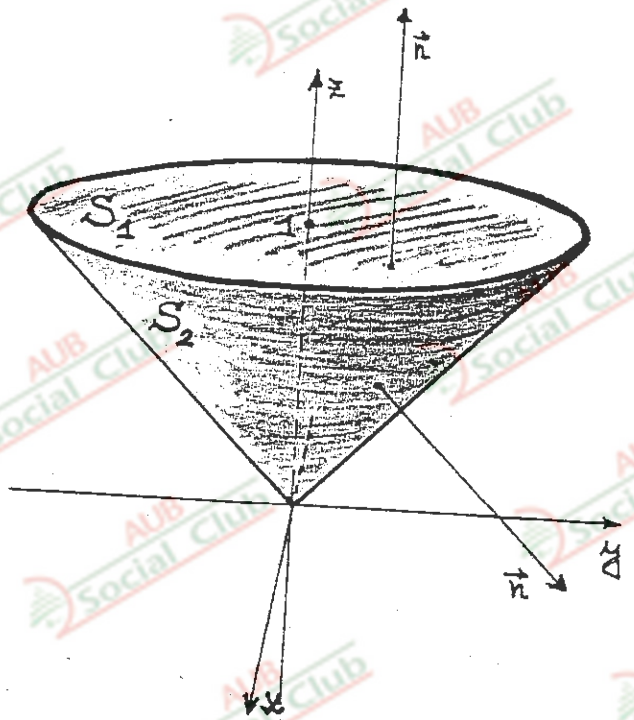
15b) Let C be the curve in the plane parametrized by

$$(x(t), y(t)) = \vec{r}(t) = (t, t^2), \qquad 0 \le t \le 2.$$

Find the work integrals $\int_C \vec{F} \cdot d\vec{r}$, $\int_C \vec{G} \cdot d\vec{r}$.

16 (12 pts). Let D be the solid cone with side $z = \sqrt{x^2 + y^2}$, top $z = 1$, and bottom $z = 0$. Let S be the surface of D (S has two parts: a flat circular lid S_1 and the conical side S_2). The normal vector on S points outwards as shown in the figure. We are also given the vector field

$$\bar{\mathrm{F}} = xz\bar{\mathrm{i}} = (xz, 0, 0).$$



16a) Explain why $\iint_{S_1} \bar{\mathrm{F}} \cdot \bar{\mathrm{n}}\, d\sigma = 0$. (This can be done without detailed calculations.)

16b) Find $\iint_{S_2} \bar{\mathrm{F}} \cdot \bar{\mathrm{n}}\, d\sigma$ by directly computing the flux integral.

16c) (Recall that the divergence theorem tells us that

$$\iint_{S_1} \bar{\mathrm{F}} \cdot \bar{\mathrm{n}}\, d\sigma + \iint_{S_2} \bar{\mathrm{F}} \cdot \bar{\mathrm{n}}\, d\sigma = \iiint_{D} (\operatorname{div} \bar{\mathrm{F}})\, dV.$$

Here if $\bar{\mathrm{F}} = M\bar{\mathrm{i}} + N\bar{\mathrm{j}} + P\bar{\mathrm{k}}$, then $\operatorname{div} \bar{\mathrm{F}} = M_x + N_y + P_z$.)

Find $\operatorname{div} \bar{\mathrm{F}}$ and integrate $\iiint_{D} (\operatorname{div} \bar{\mathrm{F}})\, dV$ directly, in order to verify that you get the same result as the sum of the answers in parts a and b above.

PLEASE START YOUR SOLUTION ON THE FOLLOWING PAGE

Math. 201 Final Exam. (Time: 140 minutes)

N. Nahlus & H. Yamani

Jan 28 , 2004

Name .... **I.D** .... **Circle *your* section number**

(Sec 9 at 1W) (sec 10 at 12:30T) (sec 11 at 2T) (sec 12 at 3:30T)

Part 1 (50 %) **10** multiple choice problems (No Penalty)

1. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1.07}}$. Then the series is

A) Conditionally convergent
B) Absolutely convergent
C) Divergent

---

2. Consider the series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3^n+5}{n!}\right)^n$. Then the series is

A) Conditionally convergent
B) Absolutely convergent
C) Divergent

---

3. Consider the series $\sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$ and $\sum_{n=2}^{\infty} (-1)^n \frac{n!\,(n+5)!}{(2n)!}$. Then

A) 1st series diverges & 2nd series diverges
B) 1st series converges & 2nd series diverges
C) 1st series diverges & 2nd series converges
D) 1st series converges & 2nd series converges

4. The domain of convergence of $\sum_{n=2}^{\infty} (-1)^n \frac{(x-1)^n}{3^n n \ln n}$ is

A) $-2 \le x < 4$ B) $-2 < x \le 4$
C) $-2 < x < 4$ D) $-2 \le x \le 4$
E) $x = 1$ & $x = 4$
F) None of the above

---

5. The Maclaurin series of $f(x) = \int_0^x \frac{1 - \cos t^{3/2}}{t} dt$ is

A) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n!} \frac{x^{3n}}{3n}$
B) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n!} \frac{x^{3n-1}}{3n-1}$
C) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n!} \frac{x^{3n+1}}{3n+1}$
D) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n!} \frac{x^{6n}}{6n}$
E) None of the above

Do not forget to integrate !

---

6. Suppose $z = f(\frac{x-y}{3y})$ where f is a differentiable function. Then $x\frac{\partial z}{\partial x} = ky\frac{\partial z}{\partial y}$ where

A) $k = -3$ B) $k = 1/3$
C) $k = 3$ D) $k = -1$
E) $k = 1$ F) $k = -1/3$
G) None of the above

7. Let $f(x, y) = e^{-3xy+5}$. Then its critical point is

A) a local max.
B) a local min.
C) a saddle point.

---

8. Consider the paraboloid $x^2 + y^2 - 4z = 1$ and the sphere $x^2 + y^2 + z^2 = 3$.
Then the ***tangent planes*** to both surfaces at the intersection point (1, 1, 1) are

A) perpendicular
B) parallel
C) neither perpendicular nor parallel.

---

9. Given that $F(x, y, z) = 8$. If the components of ∇F are never zero, then

$$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z} \quad \& \quad \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \quad \text{are}$$

A) $+1$ & $\frac{\partial z}{\partial y}$ resp. B) $+1$ & $-\frac{\partial z}{\partial y}$ resp.

C) -1 & $\frac{\partial z}{\partial y}$ resp. D) -1 & $-\frac{\partial z}{\partial y}$ resp. E) None of the above

10. The value of the double integral $\int_0^2 \int_{y/2}^1 3ye^{x^3}\,dx\,dy$ is

A) 7/2 (e-1)
B) 9/2 (e-1)
C) 2(e-1)
D) 8(e-1)
E) 25/2 (e-1)

Part II (50 %) (Subjective)

11. (5 %) Find the area of the surface cut from the bottom of the paraboloid $z = 2x^2 + 2y^2$ by the plane $z = 8$. (Grading: 4pts for setting it up & changing it to polar)

12. (7 %) Use Green's Theorem to find $\oint_C (2xy^3 + x)dx + 4x^2y^2dy$

where C (traversed counterclock wise) is the boundary of the "triangular region in the 1st quadrant enclosed by the x-axis, x=1 and $y = x^2$

---

13. (5 %) Set up (*but do not evaluate*) the double integral(s) in **polar** coordinates to find the area of the "triangular" region in the first quadrant bounded by $y=3x^2$, $x=0$ & $x+y = 4$. **Hint**: The point (1, 3) is a corner point of the region.

14. (8 %)

(i) Show that $\mathsf{F} = (x^2 - y)\mathsf{i} - (x + y^2)\mathsf{j}$ is a conservative vector field

(ii) Find a potential function for **F**

(iii) Evaluate $\oint_C (x^2 - y)\mathsf{d}\mathsf{x} - (x + y^2)\mathsf{d}\mathsf{y}$ where C is the line segment from (0, 1) to (2,0).

15. (5 %) Set up (*but do not evaluate*) the triple integral(s) in **Spherical** coordinates to find the volume and in the first octant of the surface inside the cylinder $x^2 + y^2 = 4$ and inside the sphere $x^2 + y^2 + z^2 = 29$

16. (5%) Set up (*but do not evaluate*) the triple integral(s) in **Cylinderical** coordinates to find the volume in the 1st octant common to the cylinders $x^2 + y^2 = 4$ and $x^4 + z^2 = 1$.

17. (5 %) Consider the transformation $u = x - xy$ & $v = xy$

(so $x = u + v$ & $y = \ldots\ldots$)

(i) Show that the Jacobian $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{u+v}$.

(ii) Use the above transformation to find $\iint_R x\,dydx$ where R is the region bounded by the curves $x - xy = 1$, $x - xy = 5$, $xy = 2$, $xy = 3$

18. (5 %) Use Lagrange multipliers to find the points closest to the origin on the hyperbolic cylinder $x^2 - y^2 = 1$

19. (5 %) Find **by inspection** potential functions for the following conservative fields

(i) $F = \frac{-y}{x^2+y^2}\,\mathrm{i} + \frac{x}{x^2+y^2}\,\mathrm{j}$ for (x, y) ≠ (0,0)

(ii) $F = \frac{x}{x^2+y^2}\,\mathrm{i} + \frac{y}{x^2+y^2}\,\mathrm{j}$ for (x, y) ≠ (0,0)

(iii) $F = \frac{x}{(x^2+y^2)^2}\,\mathrm{i} + \frac{y}{(x^2+y^2)^2}\,\mathrm{j}$ for (x, y) ≠ (0,0)

Box your answers

(A) (1)

Math 201–Final Exam (Fall 04) B. Shayya

- Please write your **section number** on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)
(24 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(5,0,9)}^{(1,\pi,0)} (2x\cos y + yz)\,dx + (xz - x^2\sin y)\,dy + (xy)\,dz$$

Problem 2 (answer on pages **3** and **4** of the booklet.)
(24 pts) Find the maximum and minimum values of $f(x,y,z) = xyz$ on the sphere $x^2+y^2+z^2=1$.

Problem 3 (answer on pages **5** and **6** of the booklet.)
Let D be the region bounded below by the plane $z=0$, above by the sphere $x^2+y^2+z^2=4$, and on the sides by the cylinder $x^2+y^2=1$.
(i) (8 pts) Set up the triple integrals in cylindrical coordinates that give the volume of D using the order of integration $dz\,rdr\,d\theta$. Then find the volume of D.
(ii) (6 pts) Set up the limits of integration for evaluating the integral of a function $f(x,y,z)$ over D as an iterated triple integral in the order $dy\,dz\,dx$.
(iii) (12 pts) Set up the triple integrals in spherical coordinates that give the volume of D using the order of integration $d\phi\,d\rho\,d\theta$.

Problem 4 (answer on pages **7** and **8** of the booklet.)
(25 pts) Integrate $g(x,y,z) = z$ over the surface of the prism cut from the first octant by the planes $z=x$, $z=2-x$, and $y=2$.

Problem 5 (answer on pages **9**, **10**, and **11** of the booklet.)
Let S be the cone $z = 1-\sqrt{x^2+y^2}$, $0 \le z \le 1$, and let C be its base (i.e. C is the unit circle in the xy-plane). Find the counterclockwise circulation of the field

$$F(x,y,z) = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

around C
(a) (12 pts) directly,
(b) (8 pts) using Green's theorem, and
(c) (14 pts) using Stokes' theorem (i.e. by evaluating the flux of curl F outward through S).

Problem 6 (answer on pages **12** and **13** of the booklet.)
(25 pts) Let R be the region in the xy-plane bounded by the lines $y=0$, $y=x$, $x+y=4$, and $x+y=9$. Use the transformation

$$x = uv, \qquad y = (1-u)v$$

to rewrite

$$\iint_R \frac{1}{\sqrt{x+y}}\,dx\,dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 7 (answer on page **14** of the booklet.)

(6 pts each) Determine which of the following series converge, and which diverge.

$$(a)\quad \sum_{n=1}^{\infty} \sqrt{n}\,\ln\left(1+\frac{1}{n^{2.1}}\right) \qquad (b)\quad \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{1.2}} \qquad (c)\quad \sum_{n=1}^{\infty} n(\sqrt[n]{n}-1)$$

Problem 8 (answer on pages **15** and **16** of the booklet.)

(i) (6 pts) Use Taylor's theorem to prove that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty < x < \infty).$$

(ii) (6 pts) Approximate

$$\int_0^{0.1} e^{-x^2}\,dx$$

with an error of magnitude less than 10^{-5}.

(iii) (6 pts) Show that

$$\int_0^{\infty} e^{-\pi x^2}\,dx = \frac{1}{2}.$$

(Hint. If $I = \int_0^\infty e^{-\pi x^2}dx$, then $I^2 = \int_0^\infty \int_0^\infty e^{-\pi(x^2+y^2)}dxdy$.)

(iv) (6 pts) Let E be the error resulting from the approximation

$$\int_0^{100} e^{-\pi x^2}\,dx \approx \frac{1}{2}.$$

Show that

$$|E| < \frac{e^{-5000\pi}}{2}.$$

(3)

Name: **Signature:** **Student Number:**

Mathematics 201
Final Examination, January 29, 2005, 15:00 – 17:30

This exam consists of 16 multiple choice problems and 2 workout problems.
In each multiple choice problem you must write down your choice for the answer. You will get 5 pts. if your choice is correct and 0 pts. if your choice is wrong, or when you do not write anything, or when you write more than one answer.
In each workout problem you must supply a complete solution to the problem on the page containing the problem in this booklet.
Blue booklets will **not** be graded. They are for your rough work only.
No graphing or programmable calculators are allowed during the exam.
No questions are allowed during the exam.
Do not separate pages in your booklets.

Good Luck!

Part 1: 16 multiple choice problems. Each of them is worth 5 pts. There is no penalty for wrong answers.

Problem 1. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{2^n \cdot n! \cdot n! \cdot (2x-3)^n}{(2n+1)!}$.

(a) 1 (b) 2 (c) $\frac{1}{3}$ (d) $\frac{2}{3}$
(e) none of the above

Your answer (write A, B, C, D or E):

Problem 2. Find the limit of $f(x,y) = \dfrac{e^{x+y} - e^y}{\sin(xy)}$ when $(x,y) \to (0,2)$.

(a) $-\frac{1}{2}e^2$ (b) e (c) $\frac{1}{2}e^2$ (d) the limit does not exist
(e) none of the above

Your answer (write A, B, C, D or E):

J. Nikiel

Problem 3. Decide which claim is true concerning the function $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$ and the point $P\left(-\frac{5}{3}, 0\right)$.

(a) f has a local minimum at P (b) f has a local maximum at P

(c) f has a saddle point at P (d) P is not a critical point of f

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 4. Find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$ for the transformation $x = u^2 + v^2$, $y = uv$.

(a) $-2u^2 - 2v^2$ (b) $2u + 2v$ (c) $2u^2 - 2v^2$ (d) $2u^2 + 2v^2$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 5. Evaluate $\int\int_R |x|\, dA$ where R is the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

(a) $\frac{14}{3}\pi$ (b) $2\pi \ln 2$ (c) $\frac{15}{2}\pi$ (d) $\frac{56}{3}\pi$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 6. Decide which claim is true concerning the following two series

$$[1] \sum_{n=1}^{\infty} \frac{\ln n}{n} \cdot \cos n\pi \quad \text{and} \quad [2] \sum_{n=1}^{\infty} \frac{3^n \cdot n!}{n^n} .$$

(a) [1] diverges and [2] converges (b) both series converge

(c) [1] converges and [2] diverges (d) both series diverge

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 7. The function $f(x) = \begin{cases} 1 & \text{when } -\pi < x < 0 \\ \sin x & \text{when } 0 < x < \pi \end{cases}$ is expanded into its Fourier series whose sum $F(x)$ is also the periodic extension of $f(x)$. Find $F(4\pi)$.

(a) $\frac{\pi}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 8. Evaluate the following improper integral $\int_0^2 \int_0^{\infty} (x+y) \cdot e^{-(x+y)}\, dx\, dy$.

(a) $2-5e^{-2}$ (b) $2-4e^{-1}$ (c) 2 (d) this integral diverges

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 9. Find the maximum value of $f(x,y,z) = 2x+z+2$ on the sphere $x^2+y^2+z^2 = 5$.

(a) 7 (b) $2+3\sqrt{5}$ (c) 6 (d) $7-2\sqrt{5}$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 10. Use Maclaurin series to find the derivative $f^{(2005)}(0)$ when $f(x) = x \cdot e^{-x^2}$.

(a) $\frac{2005!}{668!}$ (b) 0 (c) $\frac{2005!}{1001}$ (d) $\frac{2005!}{1002!}$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 11. Evaluate $\int_C (x+y)dx + zdy + (x+1)dz$ when C is the straight-line segment from $P_0\,(-1,1,0)$ to $P_1\,(1,2,3)$.

(a) 2 (b) $\frac{15}{2}$ (c) 10 (d) -1

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 12. Let L be the tangent line to the curve of intersection of the surfaces $2x^2+4y^2+z^2 = 10$ and $x^2+y^2+z = 4$ at the point $P\,(1,1,2)$. Then L also passes through the following point $Q\,(x,y,z)$.

(a) $(1,0,2)$ (b) $(1,3,2)$ (c) $(1,0,4)$ (d) $(0,0,0)$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 13. The following is a term of the binomial series generated by the function $f(x) = (1+x)^{\frac{3}{4}}$.

(a) $-\frac{14}{81}x^3$ (b) $\frac{5}{128}x^3$ (c) $\frac{7}{128}x^3$ (d) $-\frac{16}{81}x^3$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 14. Find $\frac{\partial z}{\partial s}$ at the point $P_0(s_0, t_0) = (0,1)$ when $z = f(x,y)$, $x = st$ and $y = s^2 + t^2$.

(a) $2\frac{\partial z}{\partial y}$ (b) $\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial y}$ (c) $\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y}$ (d) $\frac{\partial z}{\partial x}$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 15. The surface $\rho = 2\sin\phi(\cos\theta + \sin\theta)$ is a sphere. Find its radius.

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

(e) none of the above

Your answer (write A, B, C, D or E):

Problem 16. Find the volume of the region in the first octant bounded by the coordinate planes, the plane $x + z = 1$ and the surface $y = x^2 + 2z + 1$.

(a) $\frac{2}{3}$ (b) $\frac{11}{12}$ (c) $\frac{5}{12}$ (d) $\frac{7}{12}$

(e) none of the above

Your answer (write A, B, C, D or E):

Part 2: Two workout problems. Each problem is worth 10 pts.

Problem 17. Consider the vector field $\vec{F}(x,y,z) = (2x+y)\vec{i} + (x+z)\vec{j} + y\vec{k}$.

(a) Verify that this field is conservative.

(b) Find the potential function $f(x,y,z)$ for $\vec{F}$.

(c) Use the result of (b) to evaluate the integral $\int_c \vec{F} \cdot d\vec{r}$ where C is a smooth curve whose initial point is $(3,-2,1)$ and the terminal point is $(-1,2,0)$.

Your **complete** solution

– include all essential details; continue on the reverse side of this page if needed:

Problem 18. Let R be the triangular region with vertices at the points $(0,0)$, $(2,1)$ and $(1,2)$ in the xy-plane. Consider the integral $\int\int_R 4x\,dA$.

(a) Set the limits of integration in both rectangular orders $dx\,dy$ and $dy\,dx$.

(b) Set the limits of integration in polar coordinates.

(c) Evaluate the integral.

Your **complete** solution

– include all essential details; continue on the reverse side of this page if needed:

(1)

Math 201 — Fall 2004–05
Calculus and Analytic Geometry III, sections 5–8
Final exam, January 29 — Duration: 2.5 hours

YOUR NAME:

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 5	Section 6	Section 7	Section 8
Recitation M 1	Recitation Tu 12:30	Recitation Tu 2	Recitation Tu 3:30
Professor Makdisi	Mr. Khatchadourian	Mr. Khatchadourian	Mr. Khatchadourian

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READ THESE INSTRUCTIONS!!

1. PART I of the exam is quick answers. NO JUSTIFICATION REQUIRED, NO PARTIAL CREDIT.
2. PART II of the exam is short answers. No justification required, BUT PARTIAL CREDIT IS AVAILABLE.
3. PART III of the exam is full problems. FULL JUSTIFICATION AND COMPLETE SOLUTIONS ARE REQUIRED, and partial credit is available.
4. Don't forget to enter your NAME, AUB ID number, and SECTION!

GOOD LUCK!

GRADES:

1 (12 pts)	2 (6 pts)	3 (14 pts)	4 (12 pts)	5 (12 pts)	6 (16 pts)
7 (12 pts)	8 (12 pts)	9 (12 pts)	10 (12 pts)	11 (12 pts)	12 (12 pts)

TOTAL OUT OF 144:

→

PART I. QUICK ANSWERS, NO JUSTIFICATION REQUIRED, NO PARTIAL CREDIT

1 (2 pts/part, total 12 pts). Which of the following series converge and which diverge? Circle your answer.

1a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ Converges Diverges

1b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ Converges Diverges

1c) $\sum_{n=1}^{\infty} \frac{3^n + n^3}{4^n}$ Converges Diverges

1d) $\sum_{n=1}^{\infty} \sqrt[n]{n}$ Converges Diverges

1e) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\,\ln n}$ Converges Diverges

1f) $\sum_{n=1}^{\infty} \frac{\ln(1 + 1/n^{0.6})}{n^{0.6}}$ Converges Diverges

2 (2 pts/part, total 6 pts). Fill in the blanks below. (Parts a–c are not related to each other.)

2a) Fill in the blank: the sum of the following series is

$$\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n} = \underline{\qquad\qquad} .$$

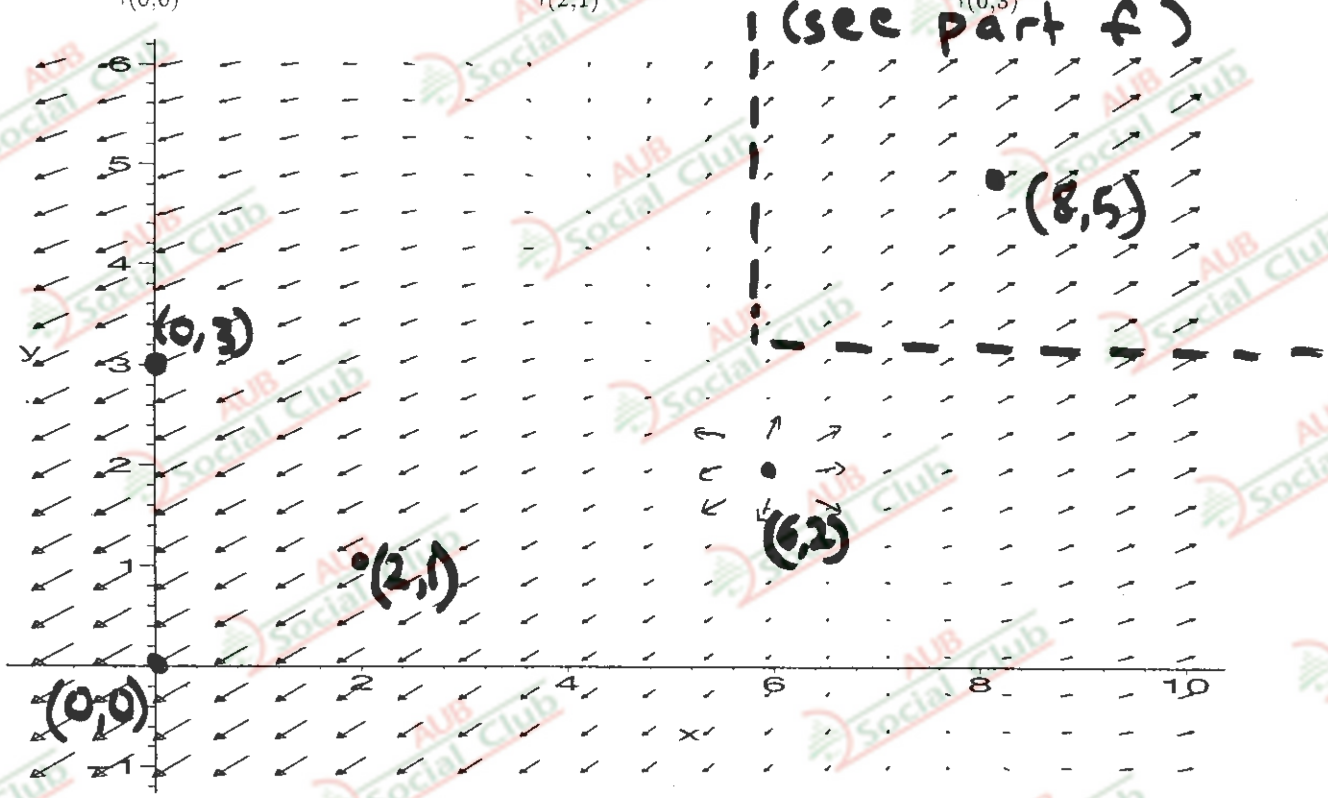
2b) Fill in the blank: $\lim_{x\to 0} \frac{\sin x - x}{x^3} = \underline{\qquad\qquad}$.

2c) Fill in the blank: the power series $\sum_{n=0}^{\infty} \frac{x^{2n}}{3^n(n^2+1)}$ has radius of convergence

$$R = \underline{\qquad\qquad} .$$

3 (total 14 pts). The following picture shows the gradient vector field $\vec{\nabla} f$ for a certain function $f(x, y)$. (Note that the gradient vectors are drawn shorter than their true length, to make things easier to visualize.) We also know the following values of the gradient at specific points:

$$\vec{\nabla} f\,\Big|_{(0,0)} = (-5, -3), \qquad \vec{\nabla} f\,\Big|_{(2,1)} = (-3, -2), \qquad \vec{\nabla} f\,\Big|_{(0,3)} = (-4, -2).$$



3a) (2 pts) Which of the following two statements is true? Circle the correct answer.

$f(0,0) > f(2,1)$ OR $f(0,0) < f(2,1)$

3b) (4 pts) Consider a moving point $P(t) = (t^2 - 2, 2t - 3)$. Fill in the blank for the derivative of $f(P(t))$ at $t = 2$:

$$\frac{d}{dt}\left[f(t^2 - 2, 2t - 3)\right]\Big|_{t=2} = \underline{\hspace{4cm}}.$$

3c) (2 pts) The point $(6, 2)$ is a critical point for f. Is it a local maximum, a local minimum, or a saddle point? Circle the correct answer.

Local maximum OR Local minimum OR Saddle point.

3d)&3e) (4 pts) Use an approximation to circle the expected answer in the two questions below:

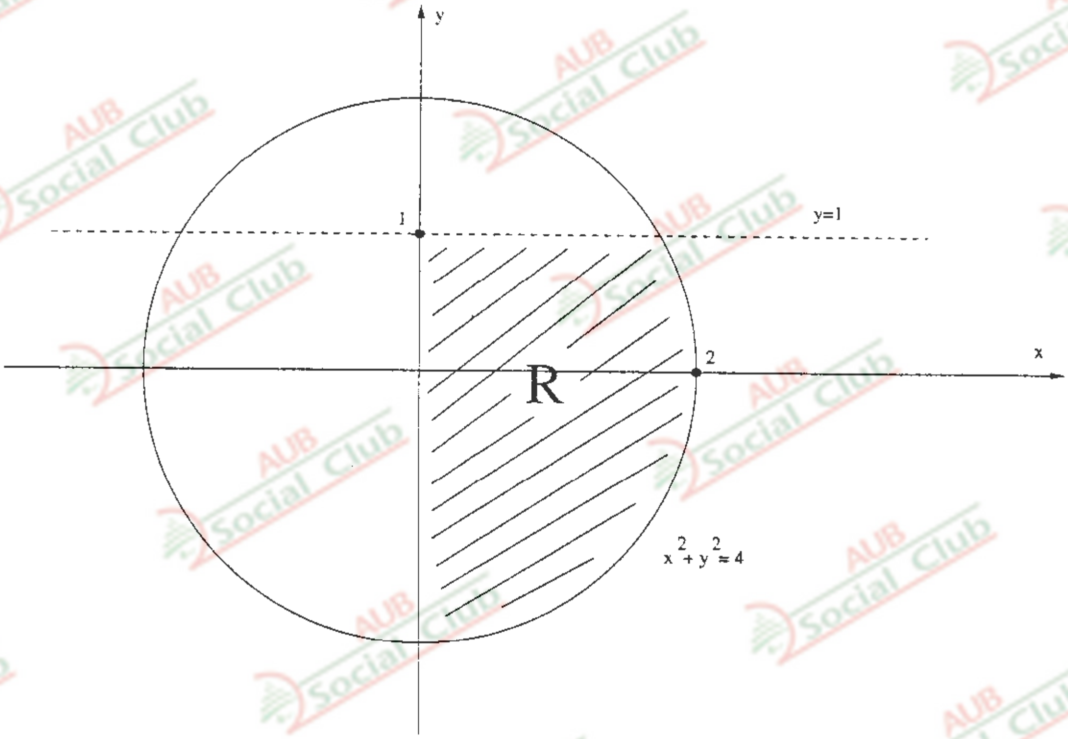
3d) The LARGEST number is : $f(0,3)$ $\quad f(0.01, 3.01)$ $\quad f(0.01, 2.99)$

3e) The SMALLEST number is : $f(0,3)$ $\quad f(0.01, 3.01)$ $\quad f(0.01, 2.99)$

3f) (2 pts) Draw a **rough sketch** of the level curve passing through the point $(8, 5)$. Please draw **ONLY** the part of the level curve that is inside the box on the top right corner, and don't clutter up the rest of the picture!

PART II. SHORT ANSWERS, no justification needed, PARTIAL CREDIT AVAILABLE.

4 (4 pts each part, total 12 pts). Consider the shaded region R below. Fill in the blanks for the following integrals in rectangular and polar coordinates:



4a) $\displaystyle\iint_R 1\,dA = \int_{y=\underline{\quad\quad}}^{\underline{\quad\quad}} \int_{x=\underline{\quad\quad}}^{\underline{\quad\quad}} dx\,dy.$

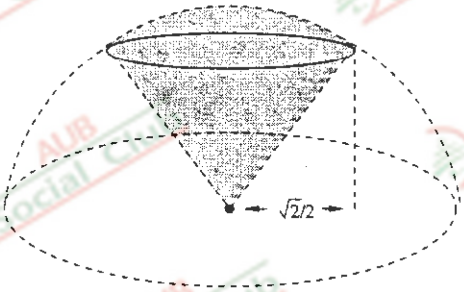
4b) $\displaystyle\iint_R 1\,dA = \int_{x=\underline{\quad\quad}}^{\underline{\quad\quad}} \int_{y=\underline{\quad\quad}}^{\underline{\quad\quad}} dy\,dx$

$\displaystyle+\int_{x=\underline{\quad\quad}}^{\underline{\quad\quad}} \int_{y=\underline{\quad\quad}}^{\underline{\quad\quad}} dy\,dx.$

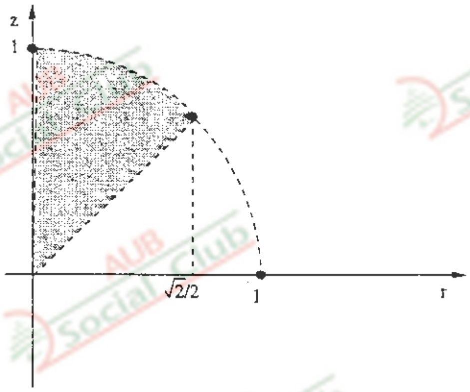
4c) $\displaystyle\iint_R 1\,dA = \int_{\theta=\underline{\quad\quad}}^{\underline{\quad\quad}} \int_{r=\underline{\quad\quad}}^{\underline{\quad\quad}} \underline{\quad\quad}\,dr\,d\theta$

$\displaystyle+\int_{\theta=\underline{\quad\quad}}^{\underline{\quad\quad}} \int_{r=\underline{\quad\quad}}^{\underline{\quad\quad}} [same]\,dr\,d\theta.$

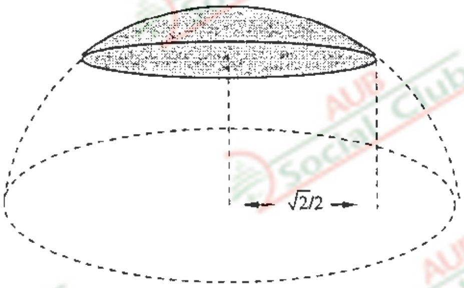
5 (4 pts each part, total 12 pts). In each of the pictures below, we give a 3-dimensional picture and a cross-section of a region D (which is always part of the half-ball $x^2+y^2+z^2 \leq 1$, $z \geq 0$). In each case, fill in the blanks for integration in spherical coordinates:



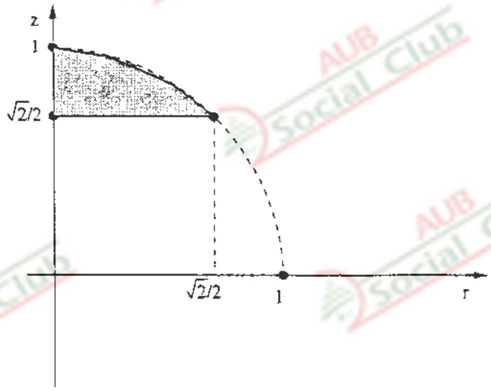
Ice–cream cone



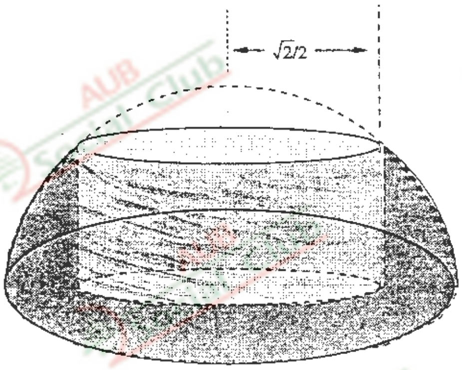
5a) $\iiint_D 1\,dV = \int_{\theta=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\rho=\_\_\_\_\_}^{\_\_\_\_\_} \_\_\_\_\_\_\_\_ \, d\rho\, d\varphi\, d\theta.$



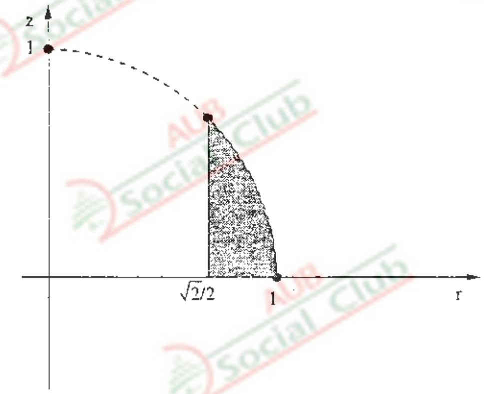
Top cut off by flat plane



5b) $\iiint_D 1\,dV = \int_{\theta=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\rho=\_\_\_\_\_}^{\_\_\_\_\_} [same] \, d\rho\, d\varphi\, d\theta.$



Hollowed–out half–ball
(cylinder of radius $\sqrt{2}/2$ removed)



5c) $\iiint_D 1\,dV = \int_{\theta=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\varphi=\_\_\_\_\_}^{\_\_\_\_\_} \int_{\rho=\_\_\_\_\_}^{\_\_\_\_\_} [same] \, d\rho\, d\varphi\, d\theta.$

6 (4 pts each part, total 16 pts). The parts are not related

6a) Find the tangent plane to the surface $x^2y + 2yz = 5$ at the point $P(1, 1, 2)$.

6b) Find a potential function for the vector field $\vec{F} = (y, x + z, y + 1)$.

6c) We make a change of variables $x = uw$, $y = u + uw$ to evaluate an integral from a region R in the xy-plane to a region R' in the uw-plane. Fill in the correct value in the blank:

$$\iint_{(x,y)\in R} 2x\, dxdy = \iint_{(u,w)\in R'} \underline{\hspace{5cm}}\, dudw.$$

6d) Given the vector field $\vec{F} = (x^2, xy, yz)$ and a surface S with oriented boundary $C = \partial S$ as shown. Then Stokes' theorem says that $\int_C \vec{F} \cdot d\vec{r} = \iint_S \vec{G} \cdot \vec{n}\, d\sigma$ for a suitable vector field $\vec{G}$. Fill in the blank:

$$\vec{G} = (\underline{\hspace{2cm}}, \underline{\hspace{2cm}}, \underline{\hspace{2cm}}).$$

PART III. FULL SOLUTIONS REQUIRED, PARTIAL CREDIT AVAILABLE.

7 (6 pts each part, total 12 pts).

a) Using power series, express the integral $L = \int_{x=0}^{0.1} e^{-x^3}\, dx$ as a certain alternating series. For full credit, the answer should be written using Σ notation. You can get nearly full credit for just writing out the first four (nonzero) terms in the series.

b) Find (with justification, of course) a specific partial sum s_n for which the error satisfies $|s_n - L| < 10^{-11}$.

Note: in parts (a) and (b), you may use without proof the fact that your series satisfies the conditions of the alternating series estimation theorem.

8 (6 pts each part, total 12 pts).

a) Use Taylor's theorem to find a specific constant A such that for all x with $|x| \leq 1/2$, we have

$$\left|(1+x)^{3/2} - 1 - 3x/2\right| \leq Ax^2. \qquad (*)$$

(You do not have to simplify your expression for A.)

b) Use the inequality $(*)$ to show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$, where $f(x,y)$ is defined by

$$f(x,y) = \frac{(1+x+y)^{3/2} - 1 - 3(x+y)/2}{\sqrt{x^2+y^2}}.$$

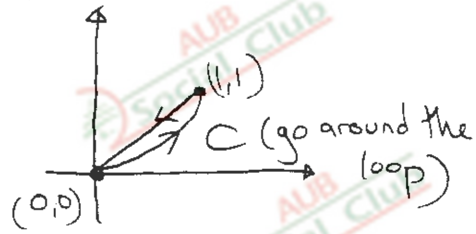
Note: you do not need to know the exact value of A to solve this part. Even if you do know A, you will have a neater solution if you just write the symbol "A" instead of its specific value each time.

9 (total 12 pts). Define the function $f(x, y) = x^2 + 2y^2 - y^3/3$.

a) (4 pts) Find and classify the critical points of f.

b) (8 pts) We **constrain** (x, y) to lie in the disk $x^2 + y^2 \leq 1$. At what points of the disk does f attain its maximum and its minimum values?

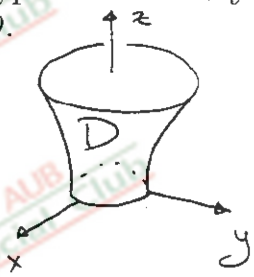
10 (6 pts each part, total 12 pts). Let C be the closed curve in the plane starting from $(0,0)$, which first goes to $(1,1)$ along the parabola $y = x^2$, and then returns to $(0,0)$ along the line $y = x$. Compute $\int_C y\,dx + e^y\,dy$ **twice**: (a) directly, (b) using Green's theorem.



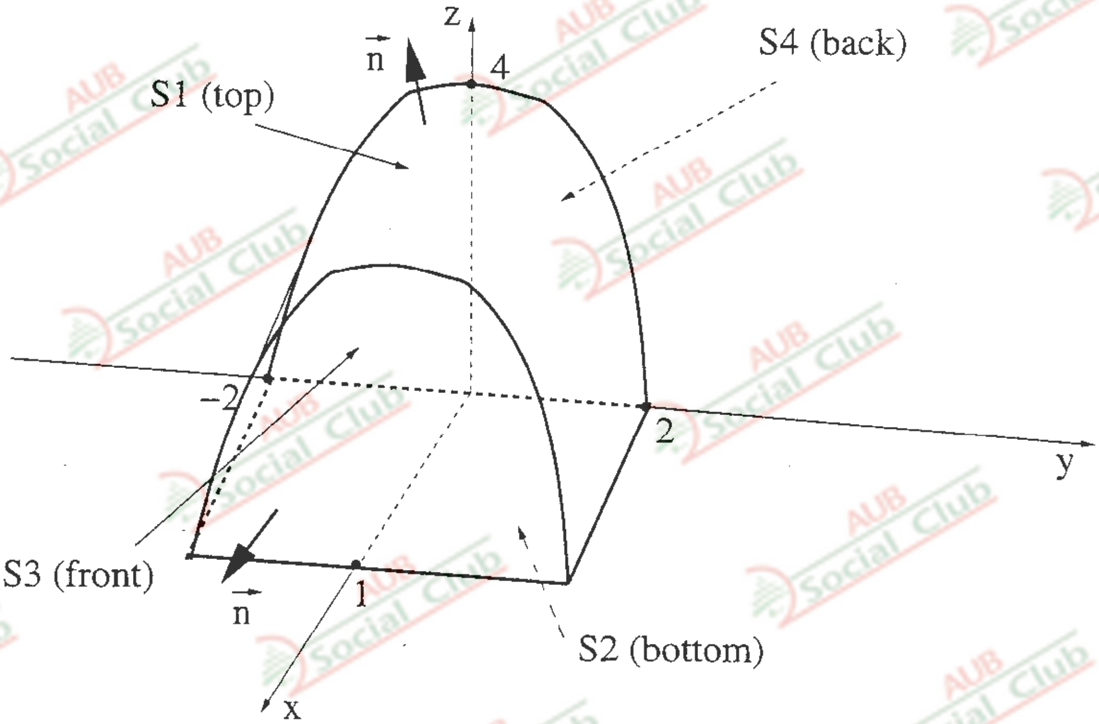
Solution for (a):

Solution for (b):

11 (12 pts). Let D be the solid region in space, shaped like a modern table, which is bounded below by the plane $z = 0$, above by the plane $z = 1$, and on the sides by the hyperboloid $x^2 + y^2 - z^2 = 1$. The density of D is $\delta(x, y, z) = 2z$. Find the total mass of D.



12 (**3 parts**, total 12 pts). Let D be the solid region in space which is bounded above by the surface $S_1 : z = 4 - y^2$, below by the plane $S_2 : z = 0$, and on the front and back by the planes $S_3 : x = 1$ and $S_4 : x = 0$, respectively. Define the vector field $\vec{F}\Big|_{(x,y,z)} = (x^2, 0, z)$. (This is the same as saying that $\vec{F} = x^2\mathbf{i} + z\mathbf{k}$.) The normal vector on the surfaces $S_1, \ldots, S_4$ is oriented outwards from D.



a) (6 pts) Compute the flux integral $\displaystyle\iint_{S_1} \vec{F} \cdot \vec{n}\, d\sigma$.

CONTINUED ON NEXT PAGE!! Parts (b) and (c) are overleaf.

Continuation of 12.

b) (2 pts) Set up but **do not evaluate** the flux integral $\iint_{S_3} \vec{F} \cdot \vec{n}\, d\sigma$.

c) (4 pts) Set up **but do not evaluate** a triple integral over the solid D which is equal by the Divergence Theorem to $\iint_{S_1} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{S_2} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{S_3} \vec{F} \cdot \vec{n}\, d\sigma + \iint_{S_4} \vec{F} \cdot \vec{n}\, d\sigma$.

(2)

FINAL EXAMINATION

MATH 201

January 29, 2005; 3:00-5:00 P.M.

Name: Signature:

Student number:

Section number (Encircle): 17 18 19 20

Instructors (Encircle): Dr. H. Yamani Mrs. M. Jurdak Prof. A. Lyzzaik

Instructions:

- No calculators are allowed.
- There are two types of questions:

PART I consists of four work-out problems. Give a detailed solution for each of these problems.

PART II consists of twelve multiple-choice questions each with **exactly one correct answer**. Circle the appropriate answer for each of these problems.

Grading policy:

- 10 points for each problem of **PART I**.
- 5 points for each problem of **PART II**.
- 0 point for no, wrong, or more than one answer of **PART II**.

GRADE OF PART I/40:

GRADE OF PART II/60:

TOTAL GRADE/100:

Part I(1). Use the change of variables $u = x - y$ and $v = x + y$ to evaluate the integral

$$\iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy$$

over the square region R bounded by the lines $x - y = 1$, $x - y = -1$, $x + y = 1$, $x + y = 3$.

Part I(2). Find the absolute maximum and minimum values of the function

$$f(x,y) = 2x^2 - xy + y^2 - 7x$$

on the square region $R = \{(x,y) : 0 \leq x, y \leq 3\}$.

Part I(3). Find the volume of the "ice cream cone" C bounded by the cone $\phi = \pi/6$ and the sphere $\rho = 2a\cos\phi$ of radius a and tangent to the $xy-$plane at the origin.

Part I(4). Use Lagrange multipliers to find the point on the plane $2x + 3y + 4z = 12$ at which the function $f(x, y, z) = 4x^2 + y^2 + 5z^2$ has its least value.

Part II

Part II(1). If $f(x,y) = \dfrac{x^4+y^2}{x^4+x^2y+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0)=0$, then

(a) f is continuous at (0,0).

(b) $\lim_{(x,y)\to(0,0)} f(x,y) = 1$.

(c) $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(d) $\lim_{(x,y)\to(0,0)} f(x,y) = 2/3$.

(e) None of the above.

Part II(2). The Maclaurin series of the integral

$$\int_0^x \sqrt{1+t^3}\, dt \quad \text{is}$$

(a) $\sum_{n=0}^{\infty} \frac{(1/2)(1/2-1)(1/2-2)\cdots(1/2-n+1)}{(3n+1)!\, n} x^{3n+1}$.

(b) $\sum_{n=0}^{\infty} \frac{(1/2)(1/2-1)(1/2-2)\cdots(1/2-n+1)}{n!\,(3n+1)} x^{3n+1}$.

(c) $\sum_{n=0}^{\infty} \frac{(1/2)(1/2-1)(1/2-2)\cdots(1/2-n+1)}{(3n+1)!} x^{3n+1}$.

(d) $\sum_{n=0}^{\infty} \frac{(1/2)(1/2-1)(1/2-2)\cdots(1/2-n+1)}{(3n+1)} x^{3n+1}$.

(e) None of the above.

Part II(3). The series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\,(\ln n)^{10}}$$

(a) converges absolutely.

(b) converges conditionally.

(c) diverges.

(d) converges conditionally and absolutely.

(e) None of the above.

Part II(4). The value of the integral

$$\int_0^1 \int_y^{\sqrt{y}} e^{y/x}\, dx\, dy \quad \text{is}$$

(a) $-1 + e/2$.

(b) $-1 + e$.

(c) $-1 + e/4$.

(d) $-1 + e/3$.

(e) None of the above.

Part II(5). The interval of convergence of the power series

$$\sum_{n=1}^{\infty} (-1)^n \frac{(x-2)^n}{n4^n} \quad \text{is}$$

(a) $[-2, 6[$.

(b) $]-2, 6[$.

(c) $[-2, 6]$.

(d) $]-2, 6]$.

(e) None of the above.

Part II(6). The value of the double integral

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{1}{1+x^2+y^2}\, dx\, dy \quad \text{is}$$

(a) $(\pi \ln 2)/2$.

(b) $\pi \ln 2$.

(c) $(\pi \ln 2)/4$.

(d) $(\pi \ln 2)/3$.

(e) None of the above.

Part II(7). The polynomial that approximates the function

$$F(x) = \int_0^x \frac{\sin t}{t}\, dt$$

throughout the interval $[0, 1/2]$ with an error of magnitude less than 10^{-3} is

(a) $x - x^3/12$.

(b) $x - x^3/6$.

(c) $x - x^3/18$.

(d) $x - x^3/24$.

(e) None of the above.

Part II(8). The series $\sum_{n=1}^{\infty} a_n$ converges if

(a) $a_n = 1/n^{\ln 2}$.

(b) $a_n = (1/n)\ln(1 + 1/n)$.

(c) $a_n < b_n$ and the series $\sum_{n=1}^{\infty} b_n$ converges.

(d) $a_n = n\sin(1/n)$.

(e) $\lim_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$.

Part II(9). Parametric equations for the line tangent to the curve of intersection of the surfaces $xyz = 1$ and $x^2 + 2y^2 + 3z^2 = 6$ at the point $P(1, 1, 1)$ are

(a) $x = 1 + t$, $y = 1 + 2t$, $z = 1 + t$, $-\infty < t < \infty$.

(b) $x = 1 - t$, $y = 1 - 2t$, $z = 1 + t$, $-\infty < t < \infty$.

(c) $x = 1 + t$, $y = 1 - 2t$, $z = 1 - t$, $-\infty < t < \infty$.

(d) $x = 1 + t$, $y = 1 - 2t$, $z = 1 + t$, $-\infty < t < \infty$.

(e) None of the above.

Part II(10). A triple integral in cylindrical coordinates for the volume of the solid cut out from the sphere $x^2 + y^2 + z^2 = 4$ by the cylinder $x^2 + y^2 = 2y$ is

(a) $\int_0^{\pi} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$.

(b) $\int_0^{\pi} \int_0^{2\sin\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$.

(c) $\int_0^{\pi/2} \int_0^{2\sin\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$.

(d) $\int_0^{\pi} \int_0^{2\sin\theta} \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$.

(e) None of the above.

Part II(11). By using Green's theorem, the value of the line integral

$$\oint_C (x+y)\, dx + (y + x^2)\, dy,$$

where C is the positively-directed boundary of the region bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, is

(a) $-\pi$.

(b) 0.

(c) -3π.

(d) -2π.

(e) None of the above.

Part II(12). The only one **TRUE** statement of the following is

(a) The field $(y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$ has potential function $xy \sin z \cos z$.

(b) The field $(e^x \cos y)\, \mathbf{i} + (e^x \sin y)\, \mathbf{j}$ is conservative.

(c) The differential $3x^2\, dx + 2xy^2\, dy$ is exact.

(d) The value of the line integral $\int_{(2,0,1/2)}^{(0,2,1)} y\, dx + x\, dy + 4\, dz$ is 2.

(e) If C is a simple closed curve, then $\oint_C y\, dx + x\, dy \neq 0$.

American University of Beirut

MATH 201

Calculus and Analytic Geometry III

Fall 2005-2006

Final Exam

Name: ....   ID #: ....

good luck

Exercise 1 *(10 points)* Given the surface $z = x^2 - 4xy + y^3 + 4y - 2$ containing the point $P(1, -1, -2)$

a. Find an equation of the tangent plane to the surface at P.

b. Find an equation of the normal line to the surface at P.

Exercise 2 *(15 points)*

a. Which of the following **series** converges and which diverges? justify.

i. $\displaystyle\sum_{n=0}^{+\infty} \frac{e^n}{1+e^{2n}}$

ii. $\displaystyle\sum_{n=1}^{+\infty} \frac{1-\cos n}{n^{\ln n}}$

iii. $\displaystyle\sum_{n=1}^{+\infty} \frac{n}{10+n^2}$

b. Find the radius of convergence of the series $\displaystyle\sum_{n=0}^{+\infty} \frac{x^{2n}}{2^n}$, then find its sum.

Exercise 3 *(10 points)*

a. If $w = f(x, y)$ is differentiable and $x = r + s, y = r - s$, show that

$$\frac{\partial w}{\partial r} \times \frac{\partial w}{\partial s} = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$$

b. Prove or disprove: The function $f(x, y) = \dfrac{x^2 y}{2x^4 + 3y^2}$ can be extended by continuity at $(0, 0)$.

Exercise 4 *(10 points)* Find the absolute minimum and maximum values of the function $f(x, y) = 4x - 8xy + 2y + 1$ on the triangular plate whose vertices are $(0, 0), (0, 1)$ and $(1, 0)$.

Exercise 5 *(10 points)* Evaluate the integral $\displaystyle\int_0^4 \int_{\sqrt{x}}^2 \sin\left(y^3\right)\, dy dx$.

Exercise 6 *(10 points)* Let $I = \int\int\int_G xyz\ dV$ where G is the solid in the first octant that is bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0, y = x$, and $y = 0$.

a. Express I as an iterated triple integral in the order $dzdydx$, then evaluate the resulting integral.

b. Express I as an iterated triple integral in the order $dxdzdy$ *(do not evaluate the integral).*

Exercise 7 *(15 points)* Rewrite the following triple integral

$$J = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{8-x^2-y^2}} z^2\ dzdxdy$$

a. in the order $dxdzdy$ *(do not evaluate the integral).*

b. in cylindrical coordinates *(do not evaluate the integral).*

c. in spherical coordinates, then evaluate the resulting integral.

Exercise 8 *(20 points)*

a. Evaluate

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz$$

b. The outward flux of a field $F = M\mathbf{i} + N\mathbf{j}$ across a simple closed curve C equals the double integral of $div\mathbf{F}$ over the region R enclosed by C:

$$\oint_C M dy - N dx = \int\int_R \Big(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\Big) dA \qquad (1)$$

Find the outward flux of the field $F = 2xy\mathbf{i} + x^2\mathbf{j}$ across the curve C in the first quadrant, bounded by the parabola $y = x^2$ and the line $y = 1$.

i. by using the line integral in the left side of equation (1)

ii. by using the double integral in the right side of equation (1)

American University of Beirut
Math 201, Fall 2005–06, sections 5–8
Final exam, February 1, 2006, 2.5 hours

Remarks and instructions:

Remember to write your **name, AUB ID number, and SECTION** on your exam booklet. The sections are: Section 5, Tu 11:00, with Ms. Jaber — Section 6, Tu 12:30, with Ms. Jaber — Section 7, Tu 2:00, with Ms. Jaber — Section 8, Tu 3:30, with Prof. Makdisi.

The exam is open book and notes. Calculators are **not** allowed. Please make it clear in your exam booklet which problem you are solving on each page. Remember to justify your work carefully.

The problems are listed in the order of the material in the book, **not** in order of increasing difficulty. **Take a few minutes to look over the exam to decide which problems you wish to work on first.** Remember to budget your time wisely. **The total number of points on the exam is 157.**

Good luck!

Question 1 (15 pts = 3 pts for each part). Which of the following series converge or diverge, and why?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{n^3+1}$ (c) $\displaystyle\sum_{n=1}^{\infty} \frac{4+\sin(n^2)}{n}$ (d) $\displaystyle\sum_{n=1}^{\infty} \cos\left[\frac{n^2}{2^n}\right]$ (e) $\displaystyle\sum_{n=1}^{\infty} \sin\left[\frac{n^2}{2^n}\right]$

Question 2 (12 pts = 6 pts for each part).

(a) Express $\displaystyle\int_{x=0}^{0.1} \ln(1+x^2)\,dx$ as a series.

(b) Find a specific partial sum s_N of the above series such that $|s_N - L| < 10^{-10}$. You do not have to check the hypotheses of the theorem that you use to estimate the error.

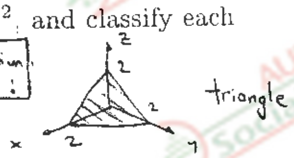
Question 3 (12 pts = 6 pts for each part).

(a) Find the third-order Taylor approximation P_3 to $f(x) = e^{2x}$ **at the center** $x = 1$. Your answer should have the form $P_3(x) = c_0 + c_1(x-1) + c_2(x-1)^2 + c_3(x-1)^3$ for specific numbers $c_0, \ldots, c_3$.

(b) As usual, we write $f(x) = P_3(x) + R_3(x)$, where R_3 is the error in the Taylor approximation. **We assume that** $0.9 \le x \le 1.1$. Find an explicit constant A for which $|R_3(x)| \le A$. (You do not need to simplify your expression for A).

Question 4 (12 pts). Find all the critical points of $f(x,y) = x^4 + 2xy + y^2$, and classify each critical point as a local maximum, a local minimum, or a saddle point.

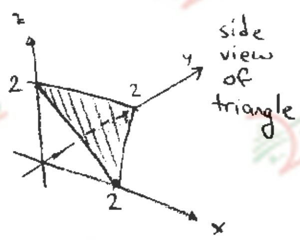
Question 5:



Question 5 (12 pts total). We wish to minimize and maximize the function $f(x,y,z) = xy^2z^3$ on the triangle given by the part of the plane $x + y + z = 2$ lying in the first octant.

(a) (2 pts) Use common sense to explain why the minimum value of f on the triangle is 0, which is attained on the edges of the triangle.

(b) (10 pts) Use Lagrange multipliers to find the maximum value of f on the triangle. Note that by part (a) you can assume that $x, y, z \neq 0$.



Question 6 (15 pts = 5 pts for each part).

(a) Given the function $f(x,y) = \sqrt{y + x^2 y}$. Find $\vec{\nabla} f$ and $\vec{\nabla} f\Big|_{(1,2)}$.

(b) Use the linear approximation (i.e., the increment theorem) to obtain an approximate value of $f(1.02, 2.01)$.

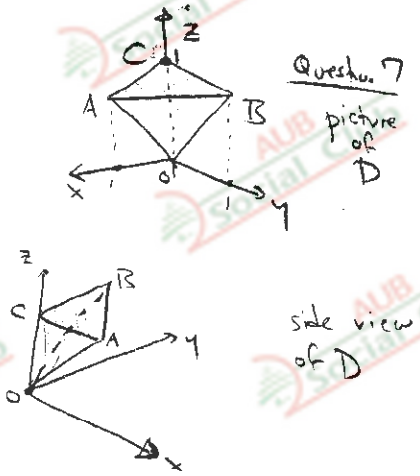
(c) Given a moving point $P(t) = (t^2 + 4t + 4, t^2 - t)$, find the value of $\dfrac{d}{dt}[f(P(t))]$ at the instant when the moving point passes through $(1,2)$.

Question 7 (20 pts total). Let D be the solid upside-down pyramid with vertices $O(0,0,0)$, $A(1,0,1)$, $B(0,1,1)$, and $C(0,0,1)$. Note that the equation of the plane OAB is $z = x+y$. The **density** of D is given by $\delta(x,y,z) = x$.

(a) (9 pts for this part) Using xyz-coordinates, set up (5 pts) and **evaluate** (4 pts) an integral in the order $\int_z \int_x \int_y$ (i.e., $dy\,dx\,dz$) that computes the total mass of D.

(b) (5 pts) Set up **but do not evaluate** the same integral in the order $\int_x \int_y \int_z$.

(c) (6 pts) Set up **but do not evaluate** the same integral in cylindrical coordinates (any order is fine, but I recommend $\int_\theta \int_r \int_z$).

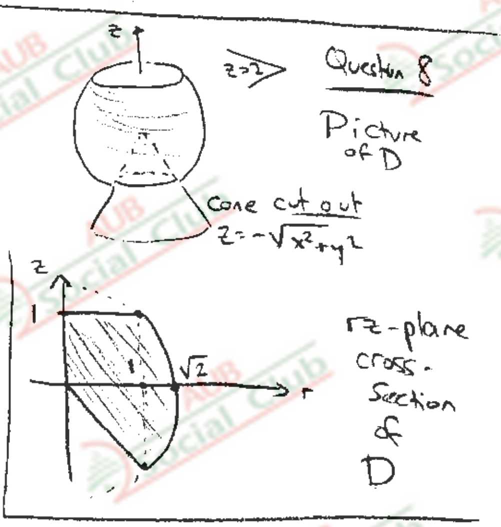


Question 8 (16 pts = 8 pts for each part). Let D be the part of the solid sphere $x^2+y^2+z^2 \le 2$ which is **below** the plane $z=1$ and **above** the cone $z = -\sqrt{x^2+y^2}$.

(a) Set up **but do not evaluate** an integral for the volume of D in cylindrical coordinates.

(b) Set up **but do not evaluate** an integral for the volume of D in spherical coordinates.

Remark: for both (a) and (b), you will need to split your integral into two parts.



Question 9 (12 pts = 6 pts for each part). (a) Find a potential function $f(x,y,z)$ for the conservative vector field $\vec{\mathbf{F}} = (y+1, x+yz^2, y^2z+z^2)$.

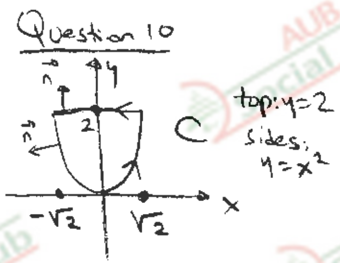
(b) Show that the vector field $\vec{\mathbf{G}} = (2xz, z^2+y^2z^3, x^2+z^2y^3)$ is **not** conservative (i.e., does not have a potential function).

Question 10 (16 pts total). Given the vector field $\vec{\mathbf{F}} = (0, 3x)$ and the closed curve C given by parts of the parabola $y = x^2$ and the line $y=2$. We orient C counterclockwise and use an outward-pointing normal.

(a) (7 pts) Directly calculate the work (i.e., circulation) integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$, by parametrizing C.

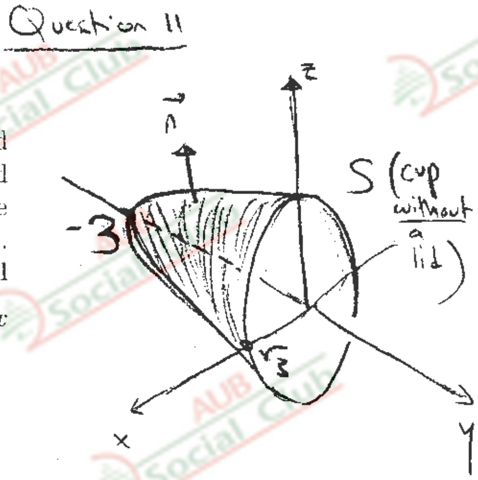
(b) (7 pts) Use Green's theorem to re-calculate $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{T}}\, ds$.

(c) (2 pts) Use Green's theorem to calculate the flux integral $\int_C \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, ds$.



Question 11 (15 pts). Given the vector field $\vec{\mathbf{F}} = (0, y, 0)$ and the surface S shaped like a cup, given by the part of the paraboloid $y = x^2+z^2-3$ cut off by the xz-plane. We orient S using the normal vector shown, which points generally away from the origin.

Set up **but do not evaluate** the **surface flux integral** $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}\, d\sigma$ in terms of an explicit integral that involves only x and z.



Math 201 Final Examination (Time: 2.5 hours) Feb 1, 2006

N. Nahlus & H. Yamani

Name

I.D Section Number

Part 1: (60/100 points:) 15 problems with 4 points each

Be Careful: Many problems have 2 parts (with 2 points each)

Circle your multiple choice answers & fill in the blanks (as in problem 1)

Penalty: – 1/2

---

1) Let $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ be the Fourier series of $f(x) = x$ on $-\pi < x < \pi$.

____ (i) Fill in the blank: $b_3 = ...\int_{....}^{...} ....dx$.

____ (ii) A) $b_3 = 0$ B) $b_3 = 2/5$ C) $b_3 = -2/5$ D) $b_3 = 1/5$ E) $b_3 = -1/5$

(F) None of the above

---

2) The series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n \ln n + 5}$ is

____ (i) X) Convergent Y) Divergent

____ (ii) A) Absolutely convergent B) Conditionally convergent C) Divergent

Reminder: Answer both parts

---

3) The series $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+2}\right)^n$ is

A) Absolutely convergent B) Conditionally convergent C) Divergent

4) The series $\sum_{n=1}^{\infty}(-1)^n \frac{(2n!)}{(n!)^2}$ is

A) Absolutely convergent B) Conditionally convergent C) Div ergent

---

5) Consider the power series $\sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^n}{\sqrt{n}}$

____(i) Fill in the blank: The domain of convergence WITHOUT ENDPOINTS

____(ii) The (full) domain of convergence is

A) $0 < x < 2$ B) $0 \le x < 2$ C) $0 \le x \le 2$

D) $0 < x \le 2$ E) x=1 F) None of the above

Reminder: Answer both parts

---

6) Consider the power series $\sum_{n=1}^{\infty} n^{2n} x^{(n^2)}$

____(i) Fill in the blank: The domain of convergence WITHOUT ENDPOINTS

____(ii) The (full) domain of convergence is

A) $-1 < x < 1$ B) $-1 \le x < 1$ C) $-1 \le x < 1$

D) $-2 < x \le 2$ E) $x = 0$ F) None of the above

7) Let $f(x) = x^7 e^{-x^2}$. Then $f^{(107)}(0) =$

A) $\dfrac{50!}{107!}$ B) $\dfrac{107!}{50!}$ C) $-\dfrac{50!}{107!}$ D) $-\dfrac{107!}{50!}$ E) None of the above

---

8a) The series $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^7}$ is A) Convergent B) Divergent

---

8b) The series $\sum_{n=2}^{\infty} \frac{1}{7^{\ln n}}$ is A) Convergent B) Divergent

---

9) Consider the paraboloid $x^2 + y^2 - 4z = -2$ and the sphere $x^2 + y^2 + z^2 = 3$.
Then the ***tangent planes*** to both surfaces at the intersection point (1, 1, 1) are

A) parallel B) perpendicular C) neither perpendicular nor parallel.

---

10) Given that $F(x, y, z) = 100$. If the components of ∇F are never zero, then

$\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y} \cdot \frac{\partial y}{\partial z}$ & $\frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial y}$ are A) 1 & $-\frac{\partial z}{\partial y}$ resp. B) -1 & $-\frac{\partial z}{\partial y}$ resp.

C) 1 & $\frac{\partial z}{\partial y}$ resp. D) -1 & $\frac{\partial z}{\partial y}$ resp. E) None of the above

11) The value of the double integral $\int_0^2 \int_{y/2}^1 6ye^{x^3}\,dx\,dy$ is

A) 9 (e-1) B) 4(e-1) C) 16(e-1) D) 25 (e-1) E) None of the above

---

12) The critical point (1, 1) of the function $f(x,y) = x^5 + y^5 - 5xy + 1$ is

A) Local Maximum B) Local Minimum C) Saddle point

---

13) The function $f(x,y)$ at the point $p(-1, 2)$ has the following directional derivavtives.
$(D_v f)(p) = 20$ & $(D_w f)(p) = 4$ where $\mathbf{v} = \mathbf{4i} + \mathbf{3j}$ & $\mathbf{w} = 7070\,\mathbf{j}$.
___(i) Fill in the blank: The gradient vector of $f(x,y)$
(Hint: Is $D_w f = f_y$?)
___(ii) The **minimum** possible directional derivative of $f(x,y)$ at p is

A) $-\sqrt{13}$ B) $-\sqrt{15}$ C) $-\sqrt{17}$ D) $-\sqrt{20}$ E) None of the above

Reminder: Answer both parts

14) Let $P = f(t^5, t^3, t^2)$ where $f(u,v,w)$ is a differentiable function.
Suppose $f(1,1,1) = 2$ & { $f_u(1,1,1) = 3$, and $f_v(1,1,1) = -5$ and $f_w(1,1,1) = 10$.}.
Then at $t = 1$, $dP/dt =$

A) 20 B) 30 C) 40 D) 50 E) None of the above

15a) $\lim_{(x,y)\to(0,0)} xy\,(\sin\frac{1}{x^2})(\cos\frac{1}{y^2})$ A) 0 B) does not exist

---

15b) $\lim_{(x,y)\to(0,0)} \frac{xy^6}{x^6 + y^4}$ A) 0 B) does not exist

---

Part 2 : (40 points) Subjective

16) (6 pts). Set up **but do not evaluate** the triple integral(s) in **cylinderical** coordinates that represent the volume of the region bounded by the paraboloids
$z = x^2 + y^2$ and $z = 18 - (x^2 + y^2)$

17) **(10 pts)** Set up **but do not evaluate** the triple integral(s) in **cylinderical & spherical** coordinates that represent the volume of the region above z=0 bounded by the right cylinder $x^2 + y^2 = 9$ and the sphere $x^2 + y^2 + z^2 = 25$

cylinderical :

spherical

$$\oint_C y^2 dx - xy dy = -3/2$$

where C is the square with vertices (0,0), (0,1), (1, 0), (1,1) (traced once and counter clockwise)

---

19) (3 pts)(i) Show that $(2xy^5 + x)dx + 5x^2y^4$ dy is an exact differential

(5 pts) (ii) Use part (i) to find (if possible) $\oint_C (2xy^5 + x)dx + 5x^2y^4\ dy$ where C is a very complicated curve from A(1,1) to B(2,5)

20) (3 pts) Find the **Jacobian** $\partial(x,y)/\partial r,q)$ for the transformation $x = r^2 q \quad y = \frac{r}{q}$

---

21) (5 pts) Find the absolute maximum & absolute minimum of the function

$$T(x,y,z) = 4x^2 + 3y^2 - 4y + 5z^2 + 20$$

on the ellipsoid $x^2 + y^2 + 5z^2 = 9$.

(Eliminate $5z^2$ OR use Lagrange method)

FINAL EXAMINATION

MATH 201

January 24, 2008; 11:30 A.M.-1:30 P.M.

Name: Signature:

Circle Your Section Number:

17. 18 19 20

24 25

Instructions:

- There are two types of questions:

PART I consists of six work-out problems. Give detailed solutions.

PART II consists of eight multiple-choice questions each with **exactly one correct answer**. Circle the appropriate answer.

Grading policy:

- 10 points for each problem of **PART I**.
- 5 points for each problem of **PART II**.
- 0 point for no, wrong, or more than one answer of **PART II**.

GRADE OF PART I	/60
GRADE OF PART II	/40
TOTAL GRADE	/100

Part I (1). Use Lagrange Multipliers to find the absolute maximum and minimum values for the function $f(x, y, z) = x - y + z$ on the unit sphere $x^2 + y^2 + z^2 - 1 = 0$.

Part I (2). Use cylindrical coordinates to find the volume of the solid bounded above by the surface $z = e^{\sqrt{x^2+y^2}}$, below by the $xy-$plane, and laterally by the cylinder $x^2 + y^2 = 1$.

Part I (3). Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 3y - 3xy$ over the triangular region R bounded by the $x-$axis, the $y-$axis, and the line $x + y = 6$.

Part I (4). Evaluate the integral

$$\iint_R \frac{2y+x}{y-2x}\, dA,$$

where R is the trapezoid with vertices $(-1,0)$, $(-2,0)$, $(0,4)$, and $(0,2)$, by using the substitution $u = y - 2x$ and $v = 2y + x$.

Part I (5). Integrate the function $f(x,y) = \sqrt{x^2+y^2}$ over the parametric curve $C: \quad \overrightarrow{\mathbf{r}}(t) = (\cos t + t\sin t)\,\overrightarrow{\mathbf{i}} + (\sin t - t\cos t)\,\overrightarrow{\mathbf{j}}, \quad 0 \le t \le \sqrt{3}$.

Part I (6). Sketch the region of integration of the integral

$$\int_0^4 \int_{\sqrt{y}}^2 x^3 \cos(xy)\, dx\, dy,$$

and evaluate the integral by reversing its order of integration.

Part II (1). If $f(x, y)$ satisfies Laplace's equation $f_{xx} + f_{yy} = 0$, then the value of the integral $\int_C f_y dx - f_x dy$ over all simple closed curves C to which Green's theorem applies is

(a) 0.

(b) 1.

(c) -1.

(d) π.

(e) $-\pi$.

Part II (2). The mass of the annular metal plate bounded by the circles $r = 1$ and $r = 2$ and whose density function is $\delta(r, \theta) = \cos^2\theta$ is

(a) $\pi/4$.

(b) $\pi/2$.

(c) $3\pi/4$.

(d) $3\pi/2$.

(e) None of the above.

Part II (3). The volume of the solid lying below the half-cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 8$ is given by the triple integral

(a) $\int_0^{2\pi} \int_{\pi/4}^{\pi} \int_0^{2\sqrt{2}} \quad \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

(b) $\int_0^{2\pi} \int_{\pi/3}^{\pi/4} \int_0^{2\sqrt{2}} \quad \rho \sin^2\phi \, d\rho \, d\phi \, d\theta$.

(c) $\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2} \quad \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

(d) $\int_0^{\pi} \int_{\pi/4}^{\pi} \int_0^{2} \quad \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$.

(e) None of the above.

Part II (4). If

$$a_n = \begin{cases} 1/\sqrt{n}; & 1 \leq n \leq 100 \\ (-1)^n/n; & n > 100, \end{cases}$$

then the series $\sum_{n=1}^{\infty} a_n$ is

(a) convergent.

(b) absolutely convergent.

(c) conditionally convergent.

(d) divergent.

(e) None of the above.

Part II (5). The power series

$$\sum_{k=2}^{\infty}(-1)^k\frac{(2x-1)^k}{k6^k}$$

has interval of **absolute** convergence

(a) $-5/2 < x \leq 7/2$.

(b) $-5/2 < x < 7/2$.

(c) $-5/2 \leq x < 7/2$.

(d) $-5/2 \leq x \leq 7/2$.

(e) None of the above.

Part II (6). The directional derivative of the function

$$f(x, y, z) = x^2 + 4y^2 - 9z^2$$

at the point $P(3, 0, -4)$ in the direction from P to the origin is

(a) 52.

(b) -52.

(c) 54.

(d) -54.

(e) None of the above.

Part II (7) The function $f(x,y) = x^2 - 4xy + y^3 + 4y$

(a) has local minimum at $(4,2)$ and local maximum at $(4/3, 2/3)$.

(b) has local minimum at $(4,2)$ and saddle point at $(4/3, 2/3)$.

(c) has local maximum at $(4,2)$ and saddle point at $(4/3, 2/3)$.

(d) has saddle point at $(4,2)$ and local minimum at $(4/3, 2/3)$.

(e) None of the above.

Part II (8) An equation of the tangent plane to the surface $2x^2 - y + 5z^2 = 0$ at the point $(1, 7, 1)$ is

(a) $3x - y + z = -3$.

(b) $x + y + 2z = 10$.

(c) $10x - y + 4z = 7$.

(d) $4x - y + 10z = 7$.

(e) None of the above.

FINAL EXAMINATION

MATH 201

January 24, 2008, 11:30-1:30pm

Name:

Student number:

Section number (Encircle): 21 22 23

Instructions:

- No calculators are allowed.
- There are two types of questions:

Part I consists of ten multiple choice questions out of 5 points each with exactly one correct answer.

Part II consists of five work-out problems out of 10 points each. Give a detailed solution for each of these problems.

Part I:

---

1 - The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{nlnn}$:

a. Converges absolutely

b. Converges conditionally

c. Diverges

---

2. The interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+\pi)^n}{\sqrt{n}}$ is:

a. $]-1-\pi, 1-\pi[$

b.$]-1-\pi, 1-\pi]$

c. $[-1-\pi, 1-\pi[$

d. $[-1-\pi, 1-\pi]$

e. None of the above

---

3. The Maclaurin series of the function $\int_1^x \frac{ln(1-t)}{t} dt$ is:

a. $-\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^2}$

b. $-\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)^2}$

c. $-\sum_{n=0}^{\infty} \frac{x^{n}}{n(n+1)}$

d. $-\sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)}$

e. None of the above.

4. The function $f(x, y, z)$ at a point P decreases most rapidly in the direction of $v = i - 2j + 3k$. In this direction the value of the derivative is $-5\sqrt{14}$. The ∇f is:

a. $-5\sqrt{14}i + 10\sqrt{14}j - 15\sqrt{14}k$

b. $5\sqrt{14}i - 10\sqrt{14}j + 15\sqrt{14}k$

c. $-5i + 10j - 15k$

d. $5i - 10j + 15k$

e. None of the above

---

5. The critical point(s) of $f(x, y) = e^{(x^2+y^2)}(2x^2 - y^2)$ is (are):

a. $(0, 0)$

b. $(0, 0), (\pm\frac{1}{\sqrt{2}}, 0)$

c. $(0, 0), (0, \pm\sqrt{2})$

d. $(0, 0), (\pm\frac{1}{\sqrt{2}}, 0), (0, \pm\sqrt{2})$

e. None of the above

---

6. Let C be a simple closed curve, then $\int_C (x+y)dx + (x^2 - y^2)dy$ is the expression of:

a. The counterclockwise circulation of the field
$F = (x^2 - y^2)i + (x + y)j$ around C.

b. The counterclockwise circulation of the field
$F = (x^2 - y^2)i - (x + y)j$ around C.

c. The outward flux of the field
$F = (x + y)i + (x^2 - y^2)j$ across C

d. The outward flux of the field
$F = (x^2 - y^2)i - (x + y)j$ across C.

e. None of the above

---

7. Use the chain rule to compute $\frac{\partial w}{\partial s}$ when $w = f(x, y, z)$,
$x = 2s + t^2, y = tu$, and $z = u^2 - s^2$:

a. $\frac{\partial w}{\partial s} = (2s + t^2)\frac{\partial w}{\partial x} + tu\frac{\partial w}{\partial y} + (u^2 - s^2)\frac{\partial w}{\partial z}$

b. $\frac{\partial w}{\partial s} = 2\frac{\partial w}{\partial x} - 2s\frac{\partial w}{\partial z}$

c. $\frac{\partial w}{\partial s} = t\frac{\partial w}{\partial y} + 2u\frac{\partial w}{\partial z}$

d. $\frac{\partial w}{\partial s} = tu\frac{\partial w}{\partial y} + (u^2 - s^2)\frac{\partial w}{\partial z}$

e. None of the above

8. The value of the double integral $\int_0^2 \int_{y/2}^1 3ye^{x^3}\, dx dy$ is:

a. $2(e-1)$

b. $\frac{9}{2}(e-1)$

c. $\frac{25}{2}(e-1)$

d. $8(e-1)$

e. None of the above

---

9. Let D be the region bounded by the paraboloids $z = 8-x^2-y^2$ and $z = x^2+y^2$. The volume of D in rectangular coordinates is expressed by the following iterated integral:

a. $\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dydxdz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dydxdz$

b. $\int_0^4 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dydxdz + \int_4^8 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dydxdz$

c. $\int_0^4 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dydxdz + \int_4^8 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dydxdz$

d. $\int_0^4 \int_{-\sqrt{z}}^{\sqrt{z}} \int_{-\sqrt{z-x^2}}^{\sqrt{z-x^2}} dydxdz + \int_4^8 \int_{-\sqrt{8-z}}^{\sqrt{8-z}} \int_{-\sqrt{8-z-x^2}}^{\sqrt{8-z-x^2}} dydxdz$

e. None of the above

---

10. The value of the integral $I = \int_0^\infty e^{-x^2} dx$ is: (Hint: Compute $I^2 = (\int_0^\infty e^{-x^2} dx)(\int_0^\infty e^{-x^2} dx)$).

a. $\frac{\pi}{4}$

b. $\frac{\sqrt{\pi}}{2}$

c. $\sqrt{\frac{\pi}{2}}$

d. $\frac{\pi}{2}$

e. None of the above

Part II

I- (10 points) Use the transformation $u = x - y$ and $v = x + y$ to evaluate the double integral:

$$\int\int_R (x-y)^2 cos^2(x+y) dA$$

over the region R bounded by the lines: $x - y = 1$, $x - y = -1$, $x + y = 1$ and $x + y = 3$.

II- (10 points) Use the method of Lagrange Multipliers to find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

III- (10 points) Let D be the region bounded by below by the xy-plane, laterally by the cylinder $x^2 + y^2 = 9$, and above by the sphere $x^2 + y^2 + z^2 = 25$. Set up, but do not evaluate, the triple integral in **cylindrical** and **spherical** coordinates that represents the volume of D.

IV- (10 points) Consider the vector field

$$F = (y + z)i + (x - 2y)j + xk$$

a- Verify that the field is conservative.
b- Find a potential function $f(x, y, z)$ for F.
c- Use part b- to evaluate the work over C where C is a smooth curve whose inital point is (3,-2,1) and the terminal point is (-1,1,2)
d- Evaluate the work by finding parametrization for the segment that make up C.

V- (10 points) Let C be the closed curve in the plane starting from $(0,0)$ which first goes to $(1,1)$ along the parabola $y = x^2$ and then returns to $(0,0)$ along the line $y = x$. Compute $\int_C y\,dx + e^y dy$ **twice**:

a) directly

b) using Green's theorem

Math 201 - Final (Fall 08)

T. Tlas

- Please answer questions 1, 2, 4, 5 and 6 on the same sheet of paper on which it is written (after the line following the question). Any part of your answer written on the wrong page will not be graded. Question 3 has an extra sheet of paper on which you can write your answer.
- When finished leave your work on your desk for it to be collected by the proctors.
- There are 6 problems in total. Most questions have several parts to them. Make sure that you attempt them all.

==

Name :

ID # :

Section Number :

==

$Q1$	
$Q2$	
$Q3$	
$Q4$	
$Q5$	
$Q6$	
$TOTAL$	

Problem 1

(10 points each) Which of the following series converge and which diverge? For those which converge, do they converge conditionally or absolutely? When possible, find the sum of the series. You should justify your answers.

i- $$\sum_{n=0}^{\infty} \frac{\cos(\pi n)}{2^n}$$

ii- $$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^e}$$

iii- $$\sum_{n=1}^{\infty} n^3 \sin\left(e^{\frac{1}{n^2}} - 1 - \frac{1}{n^2}\right)$$

iv- $$\sum_{n=1}^{\infty} \ln\left(\frac{1+\frac{1}{n}}{1+\frac{1}{n+1}}\right)$$

==

Problem 2

Consider the function

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ \pi - x & 0 \leq x \leq \pi \end{cases}$$

i- (15 points) Find the Fourier series of the function above.

ii- (10 points) Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

==

Problem 3

(40 points) Integrate the function

$$f(x,y,z) = \frac{1}{\sqrt{(x^2+y^2)(x^2+y^2+z^2)}}$$

over the region between the two spheres given by the following equations

$$\text{First Sphere}: \; x^2+y^2+z^2 = 1 \qquad \text{(unit sphere centered at the origin)}$$

$$\text{Second Sphere}: \; x^2+y^2+(z-1)^2 = 1 \qquad \text{(unit sphere centered at (0,0,1))}$$

Hint : You might find changing to spherical coordinates useful.

==

ADDITIONAL SHEET FOR PROBLEM 3 ANSWER

Problem 4

Consider the function of two variables $f(x,y) = x^3 e^y$.

i- (25 points) Find the value of the absolute maximum of this function in the region given by the following two inequalities

$$0 \leq x \leq 1 \quad , \quad 0 \leq y \leq 1$$

which is a unit square of side length equal to one, including its boundaries.

ii- (10 points) Find the line integral of the vector field

$$\vec{F}(x,y) = 3x^2 e^y \ \vec{\mathbf{i}} + x^3 e^y \ \vec{\mathbf{j}}$$

along the arc of a circle of radius one starting from the point (1,0) and going to the point (0,1). In other words find $\int_C \vec{F} \cdot d\vec{r}$, where C is the arc of the circle desribed above.

==

Problem 5

Consider the regular hexagon whose vertices are the points

$$(1\,,\,0) \quad , \quad (\frac{1}{2}\,,\,\frac{\sqrt{3}}{2}) \quad , \quad (-\frac{1}{2}\,,\,\frac{\sqrt{3}}{2}) \quad , \quad (-1\,,\,0) \quad , \quad (-\frac{1}{2}\,,\,-\frac{\sqrt{3}}{2}) \quad , \quad (\frac{1}{2}\,,\,-\frac{\sqrt{3}}{2})$$

Calculate the circulation (counterclockwise) around the hexagon of each of the following two vector fields:

i- (15 points)

$$\vec{F_1}(x,y) = x^3 \sin(x)\, e^{\cos(x^2)}\ \vec{\mathbf{i}} \ + \ \Big(1 + (y-1)^3\Big) \cos\big(e^{y\tan(y)}\big)\ \vec{\mathbf{j}}$$

ii- (15 points)

$$\vec{F_2}(x,y) = \frac{-y}{x^2+y^2}\ \vec{\mathbf{i}} \ + \ \frac{x}{x^2+y^2}\ \vec{\mathbf{j}}$$

==

Problem 6

i- (10 points) Write the Taylor series for e^{-r^4}. You do <u>not</u> have to derive it.

ii- (20 points) Evaluate the integral

$$\iiint_R \frac{e^{-(x^2+y^2)^2}}{2\pi\sqrt{x^2+y^2}}\,dV$$

with an error of no more than 0.01.

The region R is the set of all points (x, y, z) in 3 dimensions which satisfy $0 \leq z \leq 1$ and $0 \leq x^2 + y^2 \leq \frac{1}{4}$.

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American University of Beirut

MATH 201

Calculus and Analytic Geometry III

Fall 2008-2009

Final Exam

Exercise 1 a. *(5 points)* If $f(u, v, w)$ is a differentiable function and if $u = x - y$, $v = y - z$, and $w = z - x$, show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

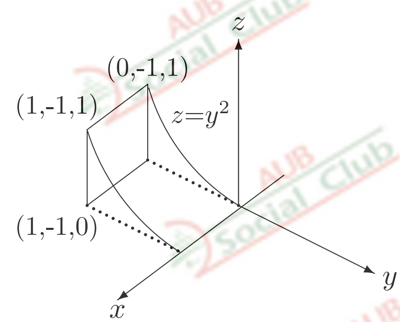
b. *(10 points)* Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y) = 3x - y + 6$ on the circle $x^2 + y^2 = 4$

Exercise 2 *(10 points)* Convert to polar coordinates, then evaluate the following integral

$$\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2 dxdy$$

Exercise 3 *(12 points)* Here is the region of integration of the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\ dy\ dx$$



Rewrite the integral as an equivalent iterated integral in the other 5 orders, then evaluate one of them

Exercise 4 Let V be the volume of the region D that is bounded by the paraboloid $z = x^2 + y^2$, and the plane $z = 2y$.

a) *(8 points)* Express V as an iterated triple integral in cartesian coordinates in the order $dz\ dx\ dy$ *(do not evaluate the integral).*

b) *(10 points)* Express V as an iterated triple integral in cylindrical coordinates, then evaluate the resulting integral.

(you may use the result: $\int \sin^4 x\ dx = -\frac{\sin^3 x \cos x}{4} - \frac{3 \cos x \sin x}{8} + \frac{3x}{8}$*)*

Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$.

a) *(7 points)* Express V as an iterated triple integral in spherical coordinates in the order $d\rho\ d\phi\ d\theta$ *(do not evaluate the integral).*

b) *(8 points)* Express V as an iterated triple integral in spherical coordinates in the order $d\phi\ d\rho\ d\theta$ *(do not evaluate the integral).*

⟶

Exercise 6 a. *(6 points)* Find the work done by the force $F = x\mathbf{i} + y^2\mathbf{j} + (y - z)\mathbf{k}$ along the straight line from $(0, 0, 0)$ to $(1, 1, 1)$.

b. *(8 points)* Evaluate

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz$$

c. Find the *outward flux* of the field $F = (y - 2x)\mathbf{i} + (x + y)\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $y = x$ and $x + y = 1$.

i. *(10 points)* by direct calculation

ii. *(6 points)* by Green's theorem

good luck

Math 201 — Fall 2008–09
Calculus and Analytic Geometry III, sections 21–23
Final exam, January 30, 2009, 2 hours 15 minutes

YOUR NAME: Questions & Solutions

YOUR AUB ID#:

PLEASE CIRCLE YOUR SECTION:

Section 21	Section 22	Section 23
Recitation M 11	Recitation M 9	Recitation M 1
Ms. Itani	Professor Makdisi	Ms. Itani

INSTRUCTIONS:

The exam is open book and notes. Calculators are **not** allowed. Remember to justify your work carefully. If you encounter a sum of fractions as a result of an integral (e.g., $1/2 + 5/17 - 7/3$), there is no need to simplify it.

As usual, you may use the back of each page for scratchwork OR for solutions. There are four extra blank sheets at the end, for extra scratchwork or solutions. If you need to continue a solution on another page, **indicate CLEARLY** where the grader should continue reading.

IMPORTANT! The problems are listed in the order of the material in the book, **not** in order of increasing difficulty. **Take a few minutes to look over the exam to decide which problems you wish to work on first.** Remember to budget your time wisely.

The total number of points on this exam is 160.

GOOD LUCK!

GRADES:

1 (20 pts)	2 (16 pts)	3 (10 pts)	4 (10 pts)	5 (16 pts)
6 (16 pts)	**7 (16 pts)**	**8 (16 pts)**	**9 (20 pts)**	**10 (20 pts)**

TOTAL OUT OF 160:

An overview of the exam problems. There are 160 points total on the exam. Take a minute to look at all the questions, THEN solve each problem on its corresponding page INSIDE the booklet.

Question 1. (4 pts each part, total 20 pts) Which of the following series converge or diverge, and why?

$$\text{(a)}\ \sum_{n=1}^{\infty} \frac{10^n}{n!} \qquad \text{(b)}\ \sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n \qquad \text{(c)}\ \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$$

$$\text{(d)}\ \sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}} \qquad \text{(e)}\ \sum_{n=1}^{\infty} \sqrt{n}\left(\frac{1}{n}-\sin\frac{1}{n}\right)$$

Question 2. (8 pts each part, total 16 pts)

(a) Find the second degree Taylor approximation P_2 to the function $f(x)=\ln x$ **at the center** $a=3$. (Your answer should have the form $P_2(x)=c_0+c_1(x-3)+c_2(x-3)^2$ for appropriate c_0, c_1, c_2.)

(b) Use the Remainder Theorem for Taylor series to show that:

$$\text{if } 2.9 \le x \le 3.1, \text{ then } \left|\ln(x)-P_2(x)\right| \le \frac{1}{72000}.$$

[Useful information so you can avoid doing long calculations by hand : $8<(2.9)^2<9$, $24<(2.9)^3<25$, $9<(3.1)^2<10$, $29<(3.1)^3<30$.]

Question 3. (10 pts) Express $\displaystyle\int_{x=0}^{0.1} \sqrt{1+x^3}\,dx$ as a series. (It is enough if you give the first four nonzero terms.)

Question 4. (10 pts) We are given a function $f(x,y,z)$ such that:

$$\vec{\nabla} f\,\Big|_{(0,1,2)} = (1,2,3), \qquad \vec{\nabla} f\,\Big|_{(1,2,3)} = (2,3,4), \qquad \vec{\nabla} f\,\Big|_{(2,3,4)} = (4,5,6).$$

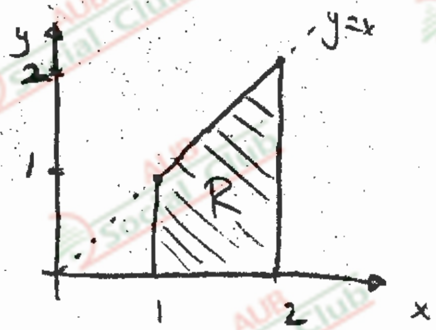
Assume that $\vec{r}(s,t)=(x(s,t),y(s,t),z(s,t))=(st, s+t, t^2)$. Find $\displaystyle\frac{\partial f(\vec{r}(s,t))}{\partial s}\Big|_{(s,t)=(1,2)}$.

Question 5. (8 pts each part, total 16 pts) Let $f(x,y)$ be the function $f(x,y)=\dfrac{x^3}{3}+y^2-2xy-8x$.

(a) Using an approximation, which do you expect to be the largest and which do you expect to be the smallest among the three numbers $f(0,1)$, $f(0.01,1.04)$, and $f(0.04,1.01)$? Explain.

(b) (Independent of part (a).) Find all the critical points of $f(x,y)$ and classify each critical point as a local maximum, a local minimum, or a saddle point.

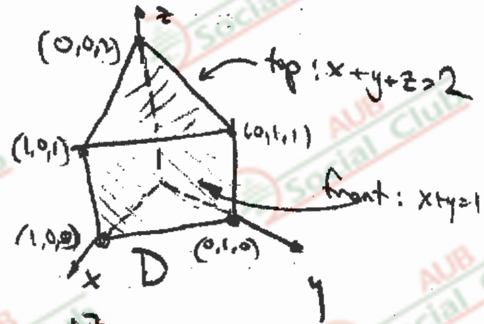
Question 6. (16 pts) Let f be the function $f(x,y,z)=(x-6)^2+y^2+2z^2$. Find the maximum and minimum of f on the solid ball $D=\{(x,y,z)\mid x^2+y^2+z^2\le 4\}$. Remember to check the interior as well as the boundary of D.



Question 7. (8 pts each part, total 16 pts) Let R be the trapezoid-shaped region in the figure, with vertices at $(1,0)$, $(2,0)$, $(1,1)$, and $(2,2)$. View R as a thin sheet with surface density $\delta(x,y) = 2xy$.

(a) Using rectangular coordinates, set up **and EVALUATE** an integral giving the total mass of R.

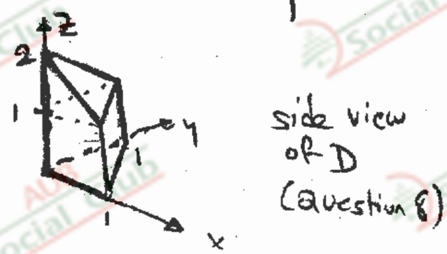
(b) Using polar coordinates, set up **but do NOT evaluate** an integral giving the total mass of R.



Question 8. (8 pts each part, 16 pts total) Let D be the region in the **first octant** cut out by the planes $x + y = 1$ and $x + y + z = 2$. See the figure.

(a) Using rectangular coordinates in the order $\int_x \int_y \int_z$ (i.e., $dz\,dy\,dx$), set up **but do not evaluate an** integral that computes the total volume of D.

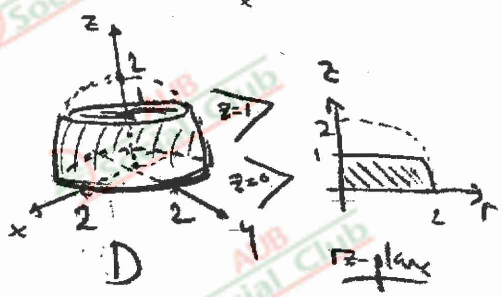
(b) Repeat part (a) using the order $\int_z \int_x \int_y$ (i.e., $dy\,dx\,dz$). You will need to split the integral in two parts. **Again, do not evaluate the integral.**



Question 9. (20 pts total) Let D be the region in space cut out from the solid ball $x^2 + y^2 + z^2 \leq 4$ by the planes $z = 0$ and $z = 1$. See the figure.

(12 pts) (a) Using cylindrical coordinates, set up **and EVALUATE** the **average value** of z on D (with respect to constant density). Use any order you find convenient for integration.

(8 pts) (b) Using spherical coordinates, set **up but do not evaluate** an integral — actually, a sum of two integrals — that gives you the **volume** of D.



Question 10. (5 pts each part, total 20 pts) We are given the following two vector fields:

$$\vec{F} = (y + 1, x + e^y), \qquad \vec{G} = (y + x, 1 + e^y).$$

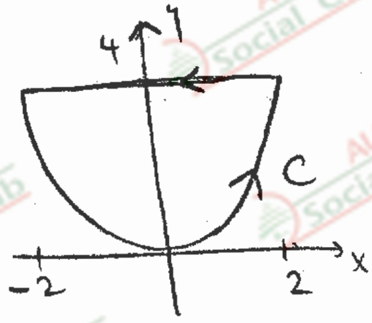
Let C be the curve shown in the figure, composed of the segment between the points $(2,4)$ and $(-2,4)$, and of part of the parabola $y = x^2$.

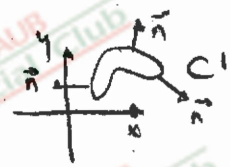
(a) Using a parametrization of (both parts of) C, compute $\int_C \vec{F} \cdot d\vec{r}$.

(b) Using Green's theorem, compute $\int_C \vec{G} \cdot d\vec{r}$.

(c) One of the fields $\vec{F}$ and $\vec{G}$ is conservative. Determine which one, and find a potential function for it.

(d) (Independent of parts (a)–(c).) Show that $\vec{G}$ has positive outward flux through any simple closed curve C': i.e., show that $\oint_{C'} \vec{G} \cdot \vec{n}\, ds > 0$.





Question 1. (4 pts each part, total 20 pts) Which of the following series converge or diverge, and why?

(a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ (b) $\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^n$ (c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}\ln n}$

(d) $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+1}}$ (e) $\sum_{n=1}^{\infty} \sqrt{n}\left(\frac{1}{n} - \sin\frac{1}{n}\right)$

a) CONVERGES: First way to see this is the ratio test:

$\rho = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left(\frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n}\right) = \lim \left(\frac{10}{n+1}\right) = 0 < 1$

Second way is by comparison: for large n, $n! >> 11^n$ (factorial growth vs. exponential growth)

so $|a_n| = \frac{10^n}{n!} << \frac{10^n}{11^n} = \left(\frac{10}{11}\right)^n$, so Σa_n converges by comparison to the geometric series $\sum_{n\geq 1}^{\infty} \left(\frac{10}{11}\right)^n$ $(r = \frac{10}{11} < 1)$.

b) DIVERGES: $a_n = (1-\frac{1}{n})^n \longrightarrow e^{-1} \neq 0$ as $n \to \infty$,

so Σa_n fails the nth term test

c) CONVERGES by the alternating series test: we have $\Sigma(-1)^n u_n$ with ① $u_n = \frac{1}{\sqrt{n}\ln n} \geq 0$; ② as n increases, $\sqrt{n}$ and $\ln n$ increase, so u_n decreases; ③ as $n \to \infty$, $u_n \to 0$ (because it is sandwiched between 0 and $\frac{1}{\sqrt{n}}$, for example).

d) DIVERGES by comparison to $\Sigma \frac{1}{n}$. (Scratch: $\frac{n}{\sqrt{n^4+1}}$ is like $\frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n}$)

careful pf: $\frac{a_n}{(\frac{1}{n})} = \frac{n}{\sqrt{n^4+1}} \cdot \frac{n}{1} = \frac{1}{\sqrt{1+\frac{1}{n^4}}} \longrightarrow 1$ as $n \to \infty$,

so for large n, $0.9 < \frac{a_n}{(\frac{1}{n})} < 1.1$, so $\frac{0.9}{n} < a_n < \frac{1.1}{n}$. But $\Sigma \frac{0.9}{n}$ diverges (harmonic series), so Σa_n diverges by comparison

e) CONVERGES by comparison test: we have $\sin u = u - \frac{u^3}{3!} + O(u^5)$ for u small,

so $\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{6n^3} + O\left(\frac{1}{n^5}\right)$ for n large, so $a_n = \sqrt{n}\left(\frac{1}{n} - \sin\frac{1}{n}\right)$ is

$\sqrt{n}\left(+\frac{1}{6n^3} + O\left(\frac{1}{n^5}\right)\right) = \frac{1}{6n^{2.5}} + O\left(\frac{1}{n^{2.5}}\right)$ & $\frac{a_n}{(1/n^{2.5})} \longrightarrow \frac{1}{6}$ as $n \to \infty$.

Thus for large n, $\frac{a_n}{(1/n^{2.5})} < 1$ (for example) i.e. $a_n < \frac{1}{n^{2.5}}$ for large n

& we compare Σa_n to the p-series $\Sigma \frac{1}{n^{2.5}}$ for $p = 2.5 > 1$.

Question 2. (8 pts each part, total 16 pts)

(a) Find the second degree Taylor approximation P_2 to the function $f(x) = \ln x$ at the center $a = 3$. (Your answer should have the form $P_2(x) = c_0 + c_1(x-3) + c_2(x-3)^2$ for appropriate c_0, c_1, c_2.)

(b) Use the Remainder Theorem for Taylor series to show that:

$$\text{if } 2.9 \le x \le 3.1, \text{ then } \left|\ln(x) - P_2(x)\right| \le \frac{1}{72000}.$$

[Useful information so you can avoid doing long calculations by hand : $8 < (2.9)^2 < 9$, $24 < (2.9)^3 < 25$, $9 < (3.1)^2 < 10$, $29 < (3.1)^3 < 30$.]

a)

n	$f^{(n)}(x)$	$f^{(n)}(3)$	$f^{(n)}(3)/n!$
0	$\ln x$	$\ln 3$	$\ln 3$
1	$1/x = x^{-1}$	$1/3$	$1/3$
2	$-x^{-2}$	$-1/9$	$-1/18$
3	$+2x^{-3}$		

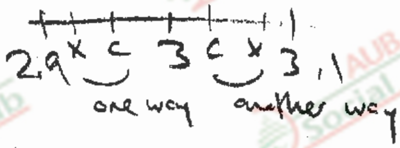
$$P_2 = \ln 3 + \frac{1}{3}(x-3) - \frac{1}{18}(x-3)^2$$

b) we have $\ln x = P_2(x) + R_2$ with $R_2 = \frac{f^{(3)}(c)}{3!}(x-3)^3 = \frac{2c^{-3}}{6}(x-3)^3$

for some c between x and 3.

We have $2.9 \le x \le 3.1$, so $|x-3| \le 0.1$

also, $2.9 \le c \le 3.1$:



So $24 < (2.9)^3 \le c^3 \le (3.1)^3 < 30$

and $\frac{1}{24} > c^{-3} > \frac{1}{30}$.

Thus $\left|\ln x - P_2(x)\right| = |R_2| = \frac{c^{-3}}{3}|x-3|^3 \le \frac{1/24}{3} \cdot (0.1)^3$

$$= \frac{1}{72} \cdot 10^{-3} = \frac{1}{72000}$$

as desired.

Question 3. (10 pts) Express $\int_{x=0}^{0.1} \sqrt{1+x^3}\,dx$ as a series. (It is enough if you give the first four nonzero terms.)

Binomial series: $\sqrt{1+u} = (1+u)^{\frac{1}{2}} = 1 + \frac{1}{2}u + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{u^2}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{u^3}{3!} + \cdots$
(for $|u|<1$)

So $\sqrt{1+x^3} = 1 + \frac{1}{2}x^3 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^6}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^9}{3!} + \cdots$

(valid for $|x^3|<1$, which is ok since we only use $|x|<0.1$)

Thus our integral is

$$\int_{x=0}^{0.1}\left(1 + \frac{1}{2}x^3 + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^6}{2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^9}{3!} + \cdots\right)dx$$

$$= \left[x + \frac{1}{2}\frac{x^4}{4} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{x^7}{7\cdot 2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{x^{10}}{10\cdot 3!} + \cdots\right]_{x=0}^{0.1}$$

$$= 0.1 + \frac{1}{2}\frac{(0.1)^4}{4} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\frac{(0.1)^7}{7\cdot 2!} + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(0.1)^{10}}{10\cdot 3!} + \cdots$$
(-0)

Question 4. (10 pts) We are given a function $f(x, y, z)$ such that:

$$\vec{\nabla} f\Big|_{(0,1,2)} = (1,2,3), \qquad \vec{\nabla} f\Big|_{(1,2,3)} = (2,3,4), \qquad \vec{\nabla} f\Big|_{(2,3,4)} = (4,5,6).$$

Assume that $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t)) = (st, s+t, t^2)$. Find $\dfrac{\partial f(\vec{r}(s,t))}{\partial s}\Big|_{(s,t)=(1,2)}$.

The chain rule says

$$\frac{\partial f(x(s,t), y(s,t), z(s,t))}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \cdot \left(\frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial z}{\partial s}\right)$$

$$= \vec{\nabla} f\Big|_{\vec{r}(s,t)} \cdot \frac{\partial \vec{r}}{\partial s}\Big|_{(s,t)}.$$

We have $\dfrac{\partial \vec{r}}{\partial s} = \dfrac{\partial}{\partial s}(st, s+t, t^2) = (t, 1, 0)$

at $(s,t) = (1,2)$, then $\vec{r}(1,2) = (1 \cdot 2, 1+2, 2^2) = (2,3,4)$

$$\frac{\partial \vec{r}}{\partial s}\Big|_{(1,2)} = (2, 1, 0)$$

$$\vec{\nabla} f\Big|_{\vec{r}(1,2)} = \vec{\nabla} f\Big|_{(2,3,4)} = (4,5,6)$$

$$\frac{\partial f(\vec{r}(s,t))}{\partial s}\Big|_{(s,t)=(1,2)} = \vec{\nabla} f\Big|_{(x,y,z)=(2,1,0)} \cdot \frac{\partial \vec{r}}{\partial s}\Big|_{(s,t)=(1,2)}$$

$$= (4,5,6) \cdot (2,1,0)$$

$$= (4)(2) + (5)(1) + (6)(0) = 8 + 5 = \boxed{13}.$$

Question 5. (8 pts each part, total 16 pts) Let $f(x,y)$ be the function $f(x,y) = \frac{x^3}{3} + y^2 - 2xy - 8x$.

(a) Using an approximation, which do you expect to be the largest and which do you expect to be the smallest among the three numbers $f(0,1)$, $f(0.01, 1.04)$, and $f(0.04, 1.01)$? Explain.

(b) (Independent of part (a).) Find all the critical points of $f(x,y)$ and classify each critical point as a local maximum, a local minimum, or a saddle point.

useful for both parts: $\vec{\nabla} f = (f_x, f_y) = (x^2 - 2y - 8,\ 2y - 2x)$.

a) $P_0(0,1)$, $P_1(0.01, 1.04)$, $P_2(0.04, 1.01)$; $\Delta\vec{r}_1 = \overrightarrow{P_0P_1} = (0.01, 0.04)$, $\Delta\vec{r}_2 = \overrightarrow{P_0P_2} = (0.04, 1.01)$

we have $\Delta f \approx \vec{\nabla} f|_{P_0} \cdot \Delta\vec{r}$ and $\vec{\nabla} f|_{(0,1)} = (0-2-8,\ 2-0) = (-10, 2)$.

$\therefore f(P_1) \approx f(P_0) + (-10,2)\cdot(0.01, 0.04)$
$= f(P_0) - 0.1 + 0.08 = f(P_0) - 0.02$

While $f(P_2) \approx f(P_0) + (-10,2)\cdot(0.04, 0.01)$
$= f(P_0) - 0.4 + 0.02 = f(P_0) - 0.38$

(the errors are $O(|\Delta\vec{r}|^2) = O(16\times10^{-4} + 10^{-4})$ so we expect ~ 3 decimals of accuracy.)

$\therefore$ $\underbrace{f(P_2)}_{\approx f(P_0) - 0.38} < \underbrace{f(P_1)}_{\approx f(P_0) - 0.02} < f(P_0)$

$f(P_2)$: smallest; $f(P_0)$: largest

(note the value of $f(P_0)$ does not affect the output; only $\vec{\nabla} f|_{P_0}$ matters).

b) critical pts occur when $\vec{\nabla} f = \vec{0} \iff \begin{cases} x^2 - 2y - 8 = 0 \\ 2y - 2x = 0 \end{cases}$ ← this says $y = x$

since $y = x$, we want $x^2 - 2x - 8 = 0$ ie $(x-4)(x+2) = 0$ so $x = 4$ or $x = -2$

but $y = x$, so our critical pts are $Q_1(4,4)$ & $Q_2(-2,-2)$

the Hessian is $H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2x & -2 \\ -2 & 2 \end{pmatrix}$

here $H|_{Q_1} = \begin{pmatrix} 8 & -2 \\ -2 & 2 \end{pmatrix}$, $D = \det H|_{Q_1} = 16 - (+4) = 12 > 0$, $f_{xx}|_{Q_1} = 8 > 0$ $\therefore Q_1$ is a local **MINIMUM**

and $H|_{Q_2} = \begin{pmatrix} -4 & -2 \\ -2 & 2 \end{pmatrix}$, $D = \det H|_{Q_2} = -8 - (+4) = -12 < 0$ $\therefore Q_2$ is a **SADDLE POINT**

Question 6. (16 pts) Let f be the function $f(x,y,z) = (x-6)^2 + y^2 + 2z^2$. Find the maximum and minimum of f on the solid ball $D = \{(x,y,z) \mid x^2+y^2+z^2 \le 4\}$. Remember to check the interior as well as the boundary of D.

The gradient of f is $\vec{\nabla} f = (2(x-6), 2y, 4z) = (2x-12, 2y, 4z)$

So the only critical point is at $P_0: (x,y,z) = (6,0,0)$.

However $P_0 \notin D$ since $6^2+0^2+0^2 > 4$. So we discard P_0

Thus f can't have a max or min on the interior of D; the max. & min. must be attained on the boundary of D which is the surface $S: x^2+y^2+z^2 = 4$ (the sphere, as opposed to the ball.)

Note moreover that D is closed & bounded, so a maximum & minimum are attained.

Thus we have reduced our problem to solving the constrained optimization problem

maximize & minimize $f = (x-6)^2 + y^2 + 2z^2$

subject to the constraint $g = x^2+y^2+z^2 = 4$

Do this problem by Lagrange multipliers: $\begin{cases} \vec{\nabla} f = \lambda \vec{\nabla} g = \lambda(2x, 2y, 2z) \\ g(P) = 4 \end{cases}$

$\Longleftrightarrow \begin{cases} 2x-12 = 2\lambda x & (1) \\ 2y = 2\lambda y & (2) \\ 4z = 2\lambda z & (3) \\ x^2+y^2+z^2 = 4 & (4) \end{cases}$

To solve this: $y \cdot (1)$ & $x \cdot (2)$ imply that $2\lambda xy = (2x-12)y = 2xy$

$\Rightarrow 2xy - 12y = 2xy \Rightarrow \boxed{y=0}$

also $z \cdot (1)$ & $x \cdot (3)$ imply that $2\lambda xz = 2xz - 12z = 4xz \Rightarrow 0 = 2xz + 12z = z(2x+12)$

Thus either $z=0$, which with $y=0$ & (4) implies $x = \pm 2 \leadsto$ 2 candidates $\begin{cases} P_1(2,0,0) \\ P_2(-2,0,0) \end{cases}$

or $2x+12=0$, & hence $x=-6$, but this is incompatible with (4) (since $x^2 \le x^2+y^2+z^2 = 4$), so no candidates occur in this case.

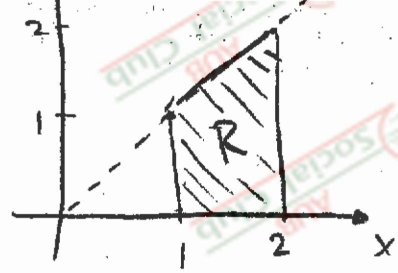
Thus the only possible candidates for maxima & minima of f on S (& hence on D as seen above) are $\begin{cases} P_1(2,0,0) \text{ with } f(P_1) = 4^2 \\ P_2(-2,0,0) \text{ with } f(P_2) = 8^2 \end{cases}$

It follows that the maximum of f on D is 8^2 $(=64)$, attained at $(-2,0,0)$ while the minimum of f on D is 4^2 $(=16)$, attained at $(2,0,0)$.

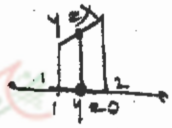
region in the figure, with vertices at (1, 0), (2, 0), (1, 1), and (2, 2). View R as a thin sheet with surface density $\delta(x, y) = 2xy$.

(a) Using rectangular coordinates, set up **and EVALUATE** an integral giving the total mass of R.

(b) Using polar coordinates, set up **but do NOT evaluate** an integral giving the total mass of R.



a) (see the figure)

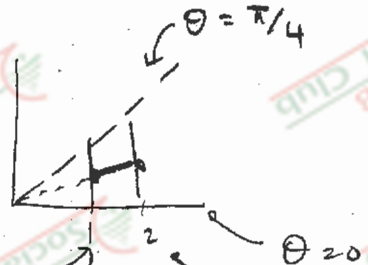


vertical slices are best here (why?)

$$\text{mass} = \iint_R \delta \, dA = \int_{x=1}^{2} \int_{y=0}^{x} 2xy \, dy \, dx = \int_{x=1}^{2} \left[xy^2 \right]_{y=0}^{x} dx$$

$$= \int_{x=1}^{2} (x^3 - 0) \, dx = \left[x^4/4 \right]_{x=1}^{2} = \frac{16-1}{4} = \boxed{15/4}.$$

b)



the line $x=1$ $\Leftrightarrow r\cos\theta = 1$ $\Leftrightarrow r = 1/\cos\theta$

similarly $r\cos\theta = 2$ $\Leftrightarrow r = \frac{2}{\cos\theta}$

we get the total mass is equal to

$$\int_{\theta=0}^{\pi/4} \int_{r=1/\cos\theta}^{2/\cos\theta} 2r^2\cos\theta\sin\theta \cdot r \, dr \, d\theta$$

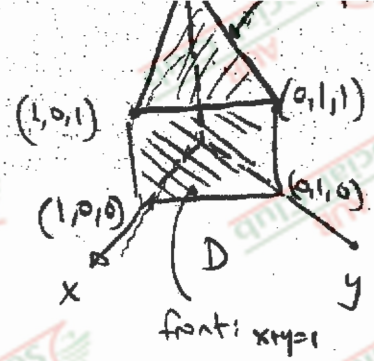
$\delta = 2xy = 2(r\cos\theta)(r\sin\theta) = 2r^2\cos\theta\sin\theta$

$dA = r \, dr \, d\theta$

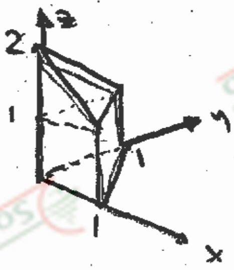
first octant cut out by the planes $x + y = 1$ and $x + y + z = 2$. See the figure.

(a) Using rectangular coordinates in the order $\int_x \int_y \int_z$ (i.e., $dzdydx$), set up **but do not evaluate** an integral that computes the total volume of D.

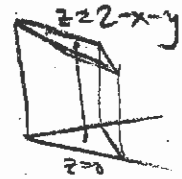
(b) Repeat part (a) using the order $\int_z \int_x \int_y$ (i.e., $dydxdz$). You will need to split the integral in two parts. **Again, do not evaluate the integral.**



side view of D:

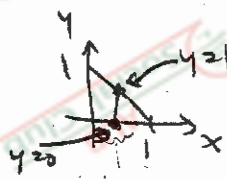


a) the projection on the xy plane is (triangle, side $x+y=1$)



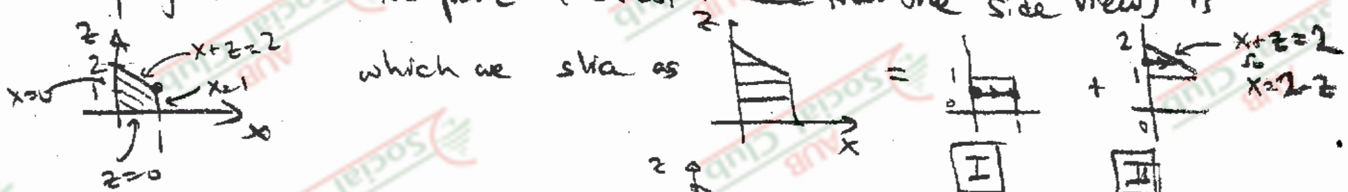
& the z-slice is from the bottom ($z=0$) to the top ($z=2-x-y$)

so we get volume $= \iiint_D 1\,dV = \iint_{(x,y)\in \triangle} \int_{z=0}^{2-x-y} dz\, dy\, dx$

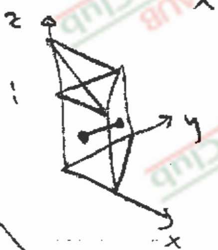


$$= \boxed{\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{2-x-y} dz\, dy\, dx}$$

b) the projection on the xz plane (easiest to see from the side view) is

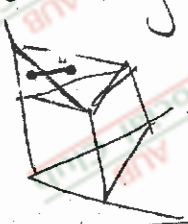


The y slice depends on the region (I or II):



in region I, the y slice starts at $y=0$ & continues to the "front" face $x+y=1$ i.e. the end is $y=1-x$

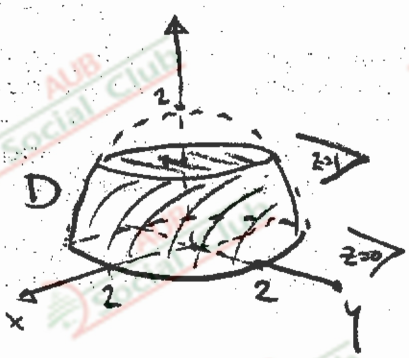
but in region II, the y-slice starts at $y=0$ & continues to the "top" (or the "roof") $x+y+z=2$ i.e. $y=2-x-z$.



so: volume of D is

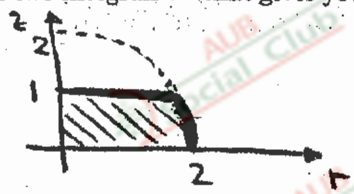
$$\underbrace{\int_{z=0}^{1} \int_{x=0}^{1} \int_{y=0}^{1-x} dy\, dx\, dz}_{\text{I}} + \underbrace{\int_{z=1}^{2} \int_{x=0}^{2-z} \int_{y=0}^{2-x-z} dy\, dx\, dz}_{\text{II}}.$$

Question 9. (20 pts total) Let D be the region in space cut out from the solid ball $x^2+y^2+z^2 \le 4$ by the planes $z=0$ and $z=1$. See the figure.



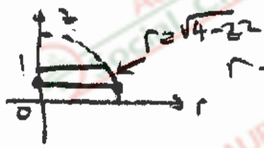
(12 pts) (a) Using cylindrical coordinates, set up and **EVALUATE** the average value of z on D (with respect to constant density). Use any order you find convenient for integration.

(8 pts) (b) Using spherical coordinates, set up **but do not evaluate** an integral — actually, a sum of two integrals — that gives you the **volume** of D.



a) the average of z is $\dfrac{\iiint_D z\,dV}{\iiint_D 1\,dV}$ $\left(=\dfrac{\iiint z\,dV}{\text{volume }(D)}\right)$.

Integrate $\iiint_D$ in cylindrical using



r-slices in the rz plane (for fixed θ)

so $\bar{z} = \dfrac{\displaystyle\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=0}^{\sqrt{4-z^2}} z\cdot \underbrace{r\,dr\,dz\,d\theta}_{dV \text{ in cylindrical}}}{\displaystyle\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\int_{r=0}^{\sqrt{4-z^2}} r\,dr\,dz\,d\theta} = \dfrac{\boxed{I}}{\boxed{II}}$

Here $\boxed{I} = \displaystyle\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\left[\frac{zr^2}{2}\right]_{r=0}^{\sqrt{4-z^2}} dz\,d\theta = \int_{\theta=0}^{2\pi}\int_{z=0}^{1}\underbrace{\left(\frac{z\cdot(4-z^2)}{2}-0\right)}_{2z-z^3/2} dz\,d\theta$

$= \displaystyle\int_{\theta=0}^{2\pi}\left[z^2-\frac{z^4}{8}\right]_{z=0}^{1} d\theta = \int_{\theta=0}^{2\pi}\frac{7}{8}\,d\theta = \frac{7}{8}\cdot 2\pi$

while $\boxed{II} = \displaystyle\int_{\theta=0}^{2\pi}\int_{z=0}^{1}\left[\frac{r^2}{2}\right]_{r=0}^{\sqrt{4-z^2}} dz\,d\theta = \int_{\theta=0}^{2\pi}\int_{z=0}^{1}\left(2-\frac{z^2}{2}\right)dz\,d\theta$

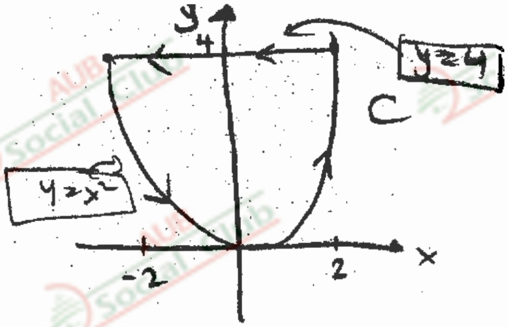
$= \displaystyle\int_{\theta=0}^{2\pi}\left[2z-\frac{z^3}{6}\right]_{z=0}^{1} d\theta = \int_{\theta=0}^{2\pi}\frac{11}{6}\,d\theta = \frac{11}{6}\cdot 2\pi$

Combining $\boxed{I}$ and $\boxed{II}$, we obtain $\bar{z} = \dfrac{7/8\cdot 2\pi}{11/6\cdot 2\pi} = \dfrac{7}{8}\times\dfrac{6}{11} = \boxed{\dfrac{21}{44}}$

note that $\bar{z}$ is slightly <u>less</u> than $\frac{1}{2}$ $\left(=\frac{22}{44}\right)$. This reflects the greater contribution of the wider bottom part of D as opposed to the thinner top part.

b) <u>see blank sheet 12.</u>

Question 10. (5 pts each part, total 20 pts) We are given the following two vector fields:



$$\vec{F} = (y+1, x+e^y), \qquad \vec{G} = (y+x, 1+e^y).$$

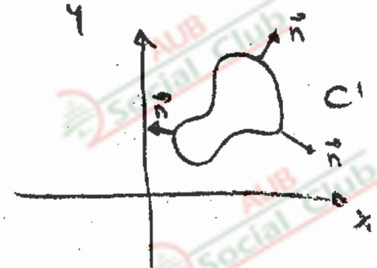
Let C be the curve shown in the figure, composed of the segment between the points $(2,4)$ and $(-2,4)$, and of part of the parabola $y = x^2$.

(a) Using a parametrization of (both parts of) C, compute $\int_C \vec{F} \cdot d\vec{r}$.

(b) Using Green's theorem, compute $\int_C \vec{G} \cdot d\vec{r}$.

(c) One of the fields $\vec{F}$ and $\vec{G}$ is conservative. Determine which one, and find a potential function for it.

(d) (Independent of parts (a)–(c).) Show that $\vec{G}$ has positive outward flux through any simple closed curve C': i.e., show that $\oint_{C'} \vec{G} \cdot \vec{n}\, ds > 0$.



a) write $C = C_1 + C_2$ where

C_1  and C_2 . Here are reasonable choices for parametrizations of C_1 & C_2: ① for C_1, $\vec{r}(t) = (t, t^2)$ with $-2 \le t \le 2$

② for C_2, $\vec{r}(t) = (-t, 4)$ with $-2 \le t \le 2$

Note the $\underline{\underline{-t}}$ in the parametrization of C_2 to get the correct orientation: $t=-2$ corresponds to $(2,4)$ & $t=+2$ corresponds to $(-2,4)$.

(N.B. other parametrizations of C_1 & C_2 are possible, for example C_2: ②': $\vec{r}(t) = (2-4t, 4)$ for $0 \le t \le 1$)

Then
$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{t=-2}^{2} (y+1, x+e^y)\Big|_{(t,t^2)} \cdot \underbrace{(1,2t)\,dt}_{d\vec{r} = \frac{d\vec{r}}{dt}dt}$$
$$+ \int_{t=-2}^{2} (y+1, x+e^y)\Big|_{(-t,4)} \cdot (-1,0)\,dt$$

$$= \int_{t=-2}^{2} (t^2+1, t+e^{t^2}) \cdot (1,2t)\,dt + \int_{t=-2}^{2} (5, -t+e^4)\cdot(-1,0)\,dt$$

$$= \int_{t=-2}^{2} \underbrace{(t^2+1+2t^2+2te^{t^2})}_{3t^2+1+2te^{t^2}}dt + \int_{t=-2}^{2} (-5+0)\,dt = \Big[t^3+t+e^{t^2}\Big]_{t=-2}^{2} + \Big[-5t\Big]_{t=-2}^{2}$$

$$= (8+2+e^4) - (-8-2+e^4) + (-10) - (+10) = 10+10-10-10 = \boxed{0}.$$

(exercise: use ②' to recalculate $\int_{C_2} \vec{F}\cdot d\vec{r}$. Also try some other parametrizations for C_1 & C_2.)

— continued on blank sheet 11 —

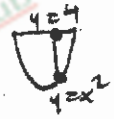
 Question 10, continued

b) $\vec{G} = (y+x,\ 1+e^y) = (M,N)$ with $\begin{cases} M = y+x \\ N = 1+e^y \end{cases}$.

since C is traversed counter-clockwise (& is a simple closed curve), Green's theorem says

$$\int_C \vec{G} \cdot d\vec{r} = \iint_R (N_x - M_y)\, dA = \iint_R (0-1)\, dA$$





$$= \int_{x=-2}^{2} \int_{y=x^2}^{4} (-1)\, dy\, dx = \int_{x=-2}^{2} (-4+x^2)\, dx = \left[-4x + \frac{x^3}{3}\right]_{x=-2}^{2}$$

$$= \left(-8 + \frac{8}{3}\right) - \left(+8 - \frac{8}{3}\right) = -8 + \frac{8}{3} - 8 + \frac{8}{3} = \boxed{\frac{-32}{3}}.$$

c) From b) $\vec{G}$ satisfies $N_x - M_y = -1 \neq 0$ so $\vec{G}$ is NOT conservative.
Alternatively, $\int_C \vec{G} \cdot d\vec{r} = -\frac{32}{3} \neq 0$ (C closed loop) so $\vec{G}$ cannot be conservative.

So $\vec{F}$ should be conservative; we want to find $f(x,y)$ such that

$\vec{\nabla} f = \vec{F} = (y+1,\ x+e^y)$. This means $\begin{cases} \partial f/\partial x = y+1 & (1) \\ \partial f/\partial y = x+e^y & (2) \end{cases}$.

Integrate (1) with respect to x (while holding y constant) to obtain

$$f = \int_{\text{"y const"}} (y+1)\, dx = xy + x + k(y) \quad (3).$$

Substitute (3) into (2) $\Rightarrow$ $x + 0 + \frac{dk}{dy} = x + e^y \Rightarrow \frac{dk}{dy} = e^y$.

$\Rightarrow k = \int e^y\, dy = e^y + C$. But we only need ONE f, so let us take $k = e^y$, so $\boxed{f = xy + x + e^y}$ is one potential function for $\vec{F}$.

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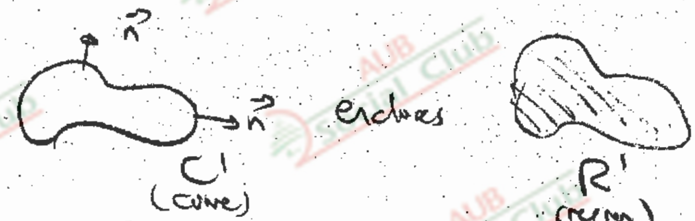
Question 10, continued



d) For $\vec{G} = (M, N)$, Green's theorem (flux form) says

$$\int_{C'} \vec{G}\cdot\vec{n}\, ds = \iint_{R'} (M_x + N_y)\, dA = \iint_{R'} (1+e^y)\, dA$$

but $M = y + x \Rightarrow M_x = 1$, $N = 1 + e^y \Rightarrow N_y = e^y$

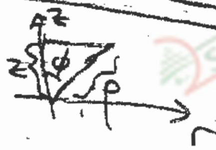
Note $1+e^y > 0$ always, so $\iint_{R'} (1+e^y)\, dA > 0$ (think of $\iint_{R'} h\, dA = \lim \left(\sum_i h(P_i)\Delta A_i\right)$, ΔA, $h(P_i) = (1+e^y)|_{P_i} > 0 \ \forall i$)

as desired.

(In fact, $1+e^y \geq 1$ so $\iint_{R'} (1+e^y)\, dA \geq \iint_{R'} 1\, dA = \text{Area}(R')$.)

Question 9, continued.

$z = 1 \Leftrightarrow \rho\cos\phi = 1$



b)

note: $\pi/3$ in the figure comes from $\cos^{-1}(1/2)$

we split up the integral into $\iiint + \iiint$

ρ sweeps in rz-plane: (for fixed ϕ)

$$\text{volume}(D) = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/3}\int_{\rho=0}^{1/\cos\phi} \underbrace{\rho^2\sin\phi\, d\rho\, d\phi\, d\theta}_{dV \text{ in spherical coordinates.}} + \int_{\theta=0}^{2\pi}\int_{\phi=\pi/3}^{\pi/2}\int_{\rho=0}^{2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2008-2009

Final Exam - solution

Exercise 1 a. If $f(u,v,w)$ is a differentiable function and if $u=x-y$, $v=y-z$, and $w=z-x$, show that $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=0$

$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x}=\frac{\partial f}{\partial u}-\frac{\partial f}{\partial w}$; $\frac{\partial f}{\partial y}=-\frac{\partial f}{\partial u}+\frac{\partial f}{\partial v}$, and $\frac{\partial f}{\partial z}=-\frac{\partial f}{\partial v}+\frac{\partial f}{\partial w}$, and hence $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}=0$

b. Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y)=3x-y+6$ on the circle $x^2+y^2=4$

$$\nabla f=\lambda\nabla g \Leftrightarrow \begin{cases} 3=2\lambda x \\ -1=2\lambda y \\ x^2+y^2=4 \quad (g(x,y)=0) \end{cases} \Leftrightarrow \begin{cases} \frac{3}{2\lambda}=x \\ -\frac{1}{2\lambda}=y \\ x^2+y^2=4 \end{cases}$$

$x^2+y^2=4 \Rightarrow \frac{9}{4\lambda^2}+\frac{1}{4\lambda^2}=4 \Rightarrow \lambda^2=\frac{10}{16} \Rightarrow \lambda=\pm\frac{\sqrt{10}}{4}$

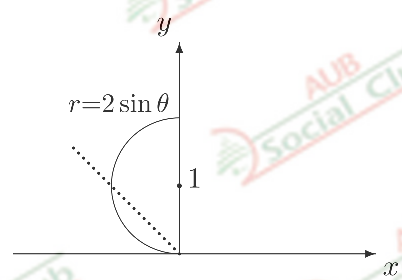
for $\lambda=\frac{\sqrt{10}}{4}$; $x=\frac{6}{\sqrt{10}}$, and $y=-\frac{2}{\sqrt{10}}$, and $f(x,y)=2\sqrt{10}+6$ (maximum value of f)

for $\lambda=-\frac{\sqrt{10}}{4}$; $x=-\frac{6}{\sqrt{10}}$, and $y=+\frac{2}{\sqrt{10}}$, and $f(x,y)=-2\sqrt{10}+6$ (minimum value of f)

Exercise 2 Convert to polar coordinates, then evaluate the following integral

$$\int_0^2\int_{-\sqrt{1-(y-1)^2}}^0 xy^2\,dx\,dy=\int_{\pi/2}^{\pi}\int_0^{2\sin\theta} r^4\cos\theta\sin^2\theta\;dr\,d\theta$$

$$=\int_{\pi/2}^{\pi}\frac{32}{5}\cos\theta\sin^7\theta\;d\theta=\frac{4}{5}\left[\sin^8\theta\right]_{\pi/2}^{\pi}=-\frac{4}{5}$$

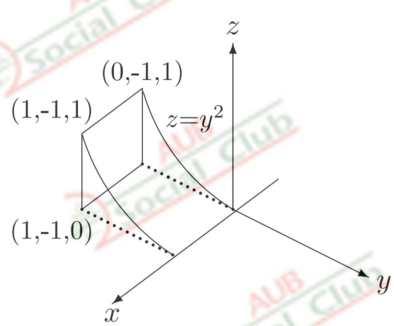


Exercise 3 $\int_0^1\int_{-1}^0\int_0^{y^2} dz\;dy\;dx=1/3$

$$\int_{-1}^0\int_0^1\int_0^{y^2} dz\;dx\;dy=\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\;dz\;dx$$

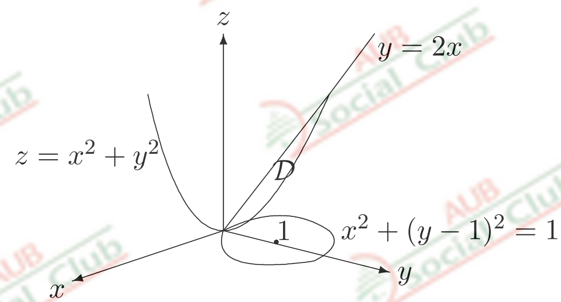
$$=\int_0^1\int_0^1\int_{-1}^{-\sqrt{z}} dy\;dx\;dz=\int_{-1}^0\int_0^{y^2}\int_0^1 dx\;dz\;dy$$

$$=\int_0^1\int_{-1}^{-\sqrt{z}}\int_0^1 dx\;dy\;dz$$



Exercise 4 Let V be the volume of the region D that is bounded by the paraboloid $z = x^2+y^2$, and the plane $z = 2y$.

a) cartesian coordinates:



$$V = \int_0^2 \int_{-\sqrt{1-(y-1)^2}}^{\sqrt{1-(y-1)^2}} \int_{x^2+y^2}^{2y} dz\, dx\, dy$$

b) cylindrical coordinates:

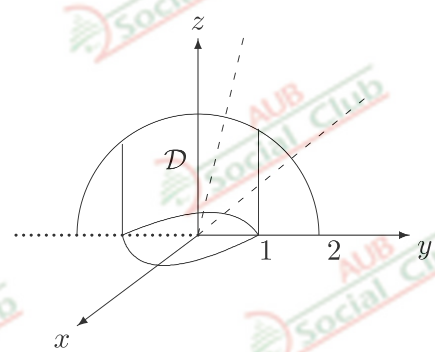
$$V = \int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} (2r^2\sin\theta - r^3)\, dr\, d\theta = \int_0^{\pi} \left[\frac{2}{3}r^3\sin\theta - \frac{r^4}{4}\right]_{r=0}^{2\sin\theta} d\theta = \int_0^{\pi} \frac{4}{3}\sin^4\theta\, d\theta$$

$$= \tfrac{4}{3}.\tfrac{3\pi}{8} = \tfrac{\pi}{2}$$

Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the sphere $x^2+y^2+z^2 = 4$, and on the sides by the cylinder $x^2+y^2 = 1$.

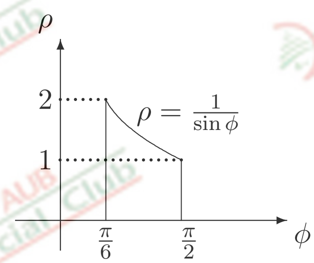
a) spherical coordinates: order $d\rho\, d\phi\, d\theta$



$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta +$$

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

b) spherical coordinates: order $d\phi\, d\rho\, d\theta$



$$V = \int_0^{2\pi} \int_0^2 \int_0^{\frac{\pi}{6}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta +$$

$$\int_0^{2\pi} \int_0^1 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta +$$

$$\int_0^{2\pi} \int_1^2 \int_{\frac{\pi}{6}}^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta$$

the answer of part b) can also be written:

$$\int_0^{2\pi} \int_0^1 \int_0^{\frac{\pi}{2}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta \quad \text{(why??)}$$

Exercise 6 a. Find the work done by the force $F = x\mathbf{i} + y^2\mathbf{j} + (y-z)\mathbf{k}$ along the straight line from $(0,0,0)$ to $(1,1,1)$.

$r(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$; $\frac{dr}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $F(t) = t\mathbf{i} + t^2\mathbf{k}$, and $F \cdot \frac{dr}{dt} = t + t^2$, hence

$$W = \int_0^1 (t+t^2)dt = 5/6$$

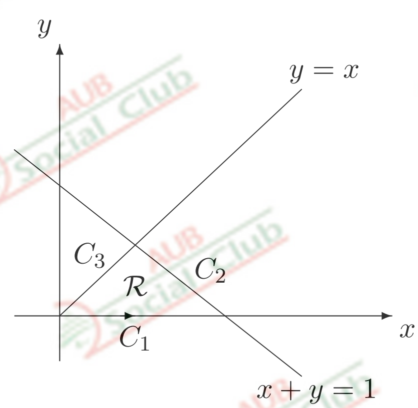
b. Evaluate

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz$$

$f(x,y,z) = x\ln z + e^x \sin y - \frac{z^2}{2} + C$ is a potential function (check it!), hence

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz = \left[x\ln z + e^x \sin y - \frac{z^2}{2}\right]_{(0,0,1)}^{(1,\pi/2,e)} = \frac{3}{2} + e - \frac{e^2}{2}$$

c. Find the *outward flux* of the field $F = (y-2x)\mathbf{i} + (x+y)\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $y = x$ and $x + y = 1$.



i) direct calculation: $Flux = \oint_C Mdy - Ndx$

$C_1 : r_1(t) = t\mathbf{i}$, $0 \le t \le 1$; $Mdy - Ndx = -tdt$

and $\int_{C_1} Mdy - Ndx = \int_0^1 -tdt = -1/2$

$C_2 : r_2(t) = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1/2$;

$Mdy - Ndx = (1-3t)dt$

and $\int_{C_2} Mdy - Ndx = \int_0^{1/2} (1-3t)dt = -1/8$

$C_3 : r_3(t) = (\frac{1}{2} - t)\mathbf{i} + (\frac{1}{2} - t)\mathbf{j}$, $0 \le t \le 1/2$; $Mdy - Ndx = (\frac{3}{2} - 3t)dt$, and

$$\int_{C_3} Mdy - Ndx = \int_0^{1/2} (\frac{3}{2} - 3t)dt = 3/8$$

$$Flux(F) = \oint_C Mdy - Ndx = -1/2 - 1/8 + 3/8 = -1/4$$

ii) Green's theorem: $Flux = \int\int_R \mathbf{divF}dA$

$$divF = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = -2 + 1 = -1$$

$$\int\int_R \mathbf{divF}dA = \int_0^{1/2}\int_y^{1-y} -\, dxdy = \int_0^{1/2} (2y-1)dy = -1/4$$

American University of Beirut

MATH 201

Calculus and Analytic Geometry III

Fall 2009-2010

Final Exam - solution

Exercise 1 a. Find the directional derivative of $f(x,y) = x^2e^{-2y}$ at $P(1,0)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

b. The equation $1 - x - y^2 - \sin(xy)$ defines y as a differentiable function of x. Find dy/dx at the point $P(0,1)$.

c. Find the points on the surface $xy + yz + zx - x - z^2 = 0$, where the tangent plane is parallel to the xy-plane.

Exercise 2 Find the absolute minimum and maximum values of $f(x,y) = x^2+xy+y^2-3x+3y$ on the triangular region R cut from the first quadrant by the line $x + y = 4$.

Exercise 3 Use Lagrange Multipliers to find the maximum and the minimum values of $f(x,y) = xy$ subject to the constraint $x^2 + y^2 = 10$.

Exercise 4 Reverse the order of integration, then evaluate the integral

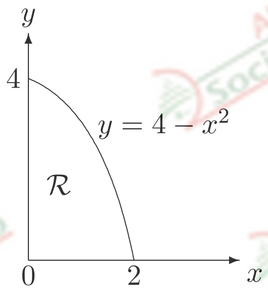
$$I = \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx$$

<u>*solution*</u>: to express the integral in the order $dxdy$, we sketch the region of integration $\mathcal{R}$ in the xy-plane.

$$I = \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$$

$$= \int_0^4 \frac{e^{2y}}{4-y} \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} dy$$

$$= \int_0^4 \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{8}\right]_0^4 = \frac{e^8-1}{4}$$



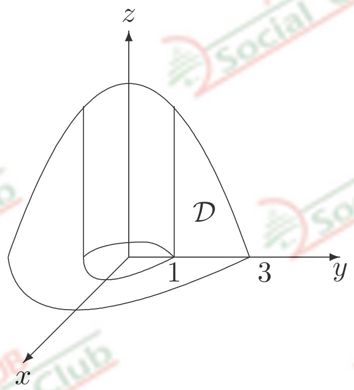
Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the paraboloid $z = 9 - x^2 - y^2$, and lying outside the cylinder $x^2 + y^2 = 1$.

<u>solution</u>:

a) $$V = \int_0^{2\pi} \int_1^3 \int_0^{9-r^2} r \; dz dr d\theta$$

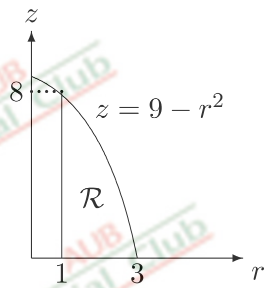
$$= 2\pi \int_1^3 r(9 - r^2)\, dr$$

$$= 2\pi \left[\tfrac{9}{2} r^2 - \tfrac{r^4}{4} \right]_1^3 = 32\pi$$



b) to express the integral in the order $drdzd\theta$, we sketch the region of integration $\mathcal{R}$ in the rz-plane.

$$V = \int_0^{2\pi} \int_0^8 \int_1^{\sqrt{9-z}} r \; dr\, dz\, d\theta$$



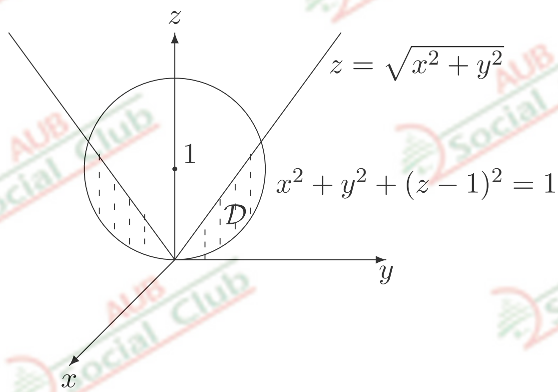
Exercise 6 Let V be the volume of the region that is bounded form below by the sphere $x^2 + y^2 + (z-1)^2 = 1$ and from above by the cone $z = \sqrt{x^2 + y^2}$. Express V as an iterated triple integral in spherical coordinates, then evaluate the resulting integral *(sketch the region of integration)*.

<u>solution</u>: The equation of the sphere $x^2 + y^2 + (z-1)^2 = 1$ in spherical coordinates is $\rho = 2\cos\phi$

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

$$= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{8}{3} \cos^3\phi \sin\phi \; d\phi \; d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} \Big[-\cos^4\phi \Big]_{\pi/4}^{\pi/2} d\theta = \int_0^{2\pi} \frac{1}{6} \; d\theta = \frac{\pi}{3}$$



Exercise 7 Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$.
(do not evaluate any of the integrals)

Exercise 8 a. Find the line integral of $f(x,y) = (x+y^2)/\sqrt{1+x^2}$ along the curve $C: y = x^2/2$ from $(1, 1/2)$ to $(0,0)$.

<u>*solution*</u>: - $r(t) = (1-t)\mathbf{i} + \frac{(1-t)^2}{2}\mathbf{j}$, $0 \le t \le 1$;

- $v(t) = \frac{dr}{dt} = -\mathbf{i} - (1-t)\mathbf{j}$, and $|v(t)| = \sqrt{1+(1-t)^2}$;

- $f(t) = \frac{(1-t)+\frac{(1-t)^4}{4}}{\sqrt{1+(1-t)^2}}$, hence

$$\int_C f(s)ds = \int_0^1 f(t).|v(t)|dt = \int_0^1 \left[(1-t) + \frac{(1-t)^4}{4}\right] dt = 11/20$$

b. Show that the differential form $2\cos y dx + (\frac{1}{y} - 2x\sin y)dy + (1/z)dz$ is exact, then evaluate

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y dx + \left(\frac{1}{y} - 2x\sin y\right) dy + (1/z)dz$$

<u>*solution*</u>: $f(x,y,z) = 2x\cos y + \ln(yz) + C$ is a potential function (check it!), hence

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y dx + \left(\frac{1}{y} - 2x\sin y\right) dy + (1/z)dz = [2x\cos y + \ln(yz) + C]_{(0,2,1)}^{(1,\pi/2,2)} = \ln(\pi/2)$$

c. Find the *counterclockwise circulation* of the field $F = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $x = 1$ and the curve $y = x^3$.

i) direct calculation: *cicculation* $= \oint_C Mdx + Ndy$

$C_1 : r_1(t) = t\mathbf{i}$, $0 \le t \le 1$; $Mdx + Ndy = 0$, and $\int_{C_1} dx + Ndy = 0$

$C_2 : r_2(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$; $Mdx + Ndy = 4t^2dt$, and $\int_{C_2} Mdx + Ndy = \int_0^1 4t^2dt = 4/3$

$C_3 : r_3(t) = (1-t)\mathbf{i} + (1-t)^3\mathbf{j}$, $0 \le t \le 1$; $Mdx + Ndy = -14(1-t)^{10}dt$, and

$$\int_{C_3} Mdx + Ndy = \int_0^1 -14(1-t)^{10}dt = -14/11$$

$$circulation(F) = \oint_C Mdx + Ndy = 0 + 4/3 - 14/11 = 2/33$$

ii) Green's theorem: $circulation(F) = \int\int_R (\mathbf{curlF}) \cdot \mathbf{k}\, dA$

$$(\mathbf{curlF}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 8xy^2 - 6xy^2 = 2xy^2$$

$$\int\int_R (\mathbf{curlF}) \cdot \mathbf{k}\, dA = \int_0^1 \int_0^{x^3} 2xy^2\, dy\, dx = \int_0^1 2x\left[\frac{y^3}{3}\right]_0^{x^3} dx = 2/3 \int_0^1 x^{10}dx = 2/33$$

American University of Beirut

Calculus and Analytic Geometry

Spring 2004

Final Exam

Date: Monday, May 31, 2004 - 3:00 pm to 5:00 pm

Instructors: Ms Sylvana Jaber, Mr Zadour Khachadourian and Dr. Mohamed Kobeissi

Name: ....

ID #: ....

Section: 4 (Ms Jaber) T 3:30-4:20 pm; 5 (Mr Khachadourian) T 2:00-2:50 pm; 6 (Mr Khachadourian) T 12:30-1:20 pm

This is **NOT** an open-book exam. Your exam should have 11 pages including this one, and there are 8 questions totaling 100 points. You can continue each exercise on the reverse side if needed.

Question	Grade
1.	
2.	
3.	
4.	
5.	
6.	
7.	
8.	

Good luck

Exercise 1 *[12 points]:* Determine if the following series converge or diverge. **Justify your answers**

a. $\displaystyle\sum_{n=1}^{+\infty} \ln\left(1 - \frac{1}{n^2}\right)$

b. $\displaystyle\sum_{n=1}^{+\infty} n \sin\frac{1}{n}$

c. $\displaystyle\sum_{n=1}^{+\infty} e^{-n}\cos n$

d. $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{\sqrt{n}\,\ln^5 n}$

Exercise 2 *[10 points]:* Evaluate the integral $\int_0^2 \int_0^1 \int_0^{1-x^2} \frac{\sin y}{\sqrt{1-y}}\, dy dx dz.$

Exercise 3 *[16 points]*: Let V be the volume of the smaller region cut from the bottom of the cone $z = \sqrt{x^2 + y^2}$ by the plane $z = 3$.

a. Express V as an iterated triple integral in spherical coordinates (that is, set up the limits of integration but do not evaluate the resulting integral).

b. Express V as an iterated triple integral in cylindrical coordinates, then evaluate the resulting integral to find V.

c. Find the plane $z = c$ that divides the region into two parts of equal volume.

Exercise 4 *[14 points]:* Find the absolute minimum and maximum for the function $f(x, y) = x^2 + 4y^2 - y$ on the region $R = \{(x, y): x^2 + 4y^2 \leq 1, y \geq 0\}$.

Exercise 5 *[10 points]:* Let V be the volume of the region whose vertices are $(0,0,0)$, $(1,2,0)$, $(1,2,1)$, $(0,2,0)$ and $(0,2,1)$.

Express V as an iterated triple integrals in the following orders

a. order dzdxdy (do not evaluate the integral)

b. order dydzdx, then evaluate the integral

Exercise 6 *[12 points]:* Evaluate the integral

$$\int_0^1 \int_y^{2-y} e^{(x-y)/(x+y)}\, dx dy$$

by applying the transformation $u = x - y$ and $v = x + y$.

Exercise 7 *[16 points]:*

a. Find the value of the line integral

$$\int_{(1,0,0)}^{(2,\pi/2,1)} (3x^2 + 2xz^2)dx + (7z\cos y)dy + (2x^2z + 7\sin y + e^z)dz$$

b. Find the counterclockwise circulation of the field $F(x, y) = (x + e^x \sin y)\mathbf{i} + (x + e^x \cos y)\mathbf{j}$ around the triangle whose vertices are (0,0), (1,0) and (0,1)

Exercise 8 *[10 points]:*

a. Show that $\lim_{(x,y)\to(0,0)} \cos\left(\frac{\pi x y^2}{x^2+y^4}\right)$ does not exist.

b. Let $\omega = f(x, y)$ be a differentiable function where $x = r/s$ and $y = s/r$. Use the chain rule to find $\frac{\partial \omega}{\partial r}$ at $(r, s) = (1, 2)$.

American University of Beirut

MATH 201

Calculus and Analytic Geometry III

Spring 2005

Final Exam

Name: .... ID #: ....

Exercise 1 *[9 points]* Discuss whether the following series converges or diverges:

a) $\sum_{n=1}^{+\infty} \frac{\sqrt[n]{n}}{1+n^2}$

b) $\sum_{n=1}^{+\infty} \frac{1.3.5.7...(2n-1)}{(2n)!}$

c) $\sum_{n=0}^{+\infty} \frac{e^n}{3^{n(n+1)}}$

Exercise 2 *[10 points]* Find the interval of convergence of the power series $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^2+1}} (x-\frac{1}{2})^n$

(be sure to check at the end points)

Exercise 3 a) *[5 points]* Find an equation of the tangent plane and a parametric equation for the normal line to the surface $z = 3x^2y - 5x + 1$ at the point $(2,1,3)$.

b) *[6 points]* Suppose that $z = e^{xy}$ and that $x = 2u + v, y = u/v$. Use the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

(give the expressions in terms in u and v)

c) *[5 points]* Prove or disprove: $g(x,y) = \frac{x^2y^2}{x^4+3y^4}$ can be extended by continuity at $(0,0)$. Justify.

Exercise 4 *[15 points]* Find the absolute minimum and maximum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.

Exercise 5 *[10 points]* Evaluate the double integral

$$\int_0^2 \int_0^{\sqrt{1-(y-1)^2}} \frac{x}{x^2+y^2}\, dx dy$$

by converting it to a double integral in polar coordinates.

Turn over →

Exercise 6 *[10 points]* Let V be the volume of the tetrahedron bounded by the coordinates planes and the plane $z = 4-4x-2y$. Express V as iterated triple integral in cartesian coordinates in the order:

a) $dxdzdy$ *(do not evaluate the resulting integral).*

b) $dzdydx$, then evaluate the resulting integral.

Exercise 7 *[15 points]* Let V be the volume of the region R cut from the cone $z = \sqrt{x^2+y^2}$ by the two planes $z = 1$ and $z = 2$.

a) Sketch the region of integration.

b) Express V as iterated triple integral in cylindrical coordinates *(do not evaluate the resulting integral).*

c) Express V as iterated triple integral in spherical coordinates, then evaluate the resulting integral.

Exercise 8 *[15 points]*

a) Find the work done by the force field defined by

$$F(x,y,z) = 4y\mathbf{i} + 2xz\mathbf{j} + 3y\mathbf{k}$$

acting on an abject as it moves along the line segment from $(1,1,1)$ to $(2,2,1)$.

b) Evaluate the line integral $\oint_{(3,-2,0)}^{(1,0,\pi)} (2x\cos z - x^2)dx + (z-2y)dy + (y - x^2\sin z)dz$.

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2007-2008

Final Exam

Name: .... ID #: ....

Exercise 1 *(12 points)* Discuss the convergence of the following series:

$$a)\ \sum_{n=0}^{+\infty}\frac{1}{\sqrt{n!}} \qquad b)\ \sum_{n=0}^{+\infty}\frac{n+10}{n\ln^3 n} \qquad c)\ \sum_{n=0}^{+\infty}(-1)^{n^2}\frac{1}{n^2+\sqrt{n}}$$

Exercise 2 *(10 points)* If $w=f(x,y)$ is differentiable and $x=r+s, y=r-s$, show that

$$\frac{\partial w}{\partial r}\times\frac{\partial w}{\partial s}=\left(\frac{\partial f}{\partial x}\right)^2-\left(\frac{\partial f}{\partial y}\right)^2$$

Exercise 3 *(15 points)* Find the absolute extrema of $f(x,y)=5+4x-2x^2+3y-y^2$ on the region R bounded by the lines $y=2, y=x$, and $y=-x$.

Exercise 4 *(8 points)* Use Lagrange multipliers to find the maximum and minimum of $f(x,y)=4xy$ subject to $x^2+y^2=8$.

Exercise 5 *(15 points)* Evaluate the integral $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^x\frac{\sin(2z)}{4-z}\,dydzdx$

Exercise 6 *(10 points)*
Evaluate the integral $\displaystyle\int\int_D\cos(x^2+y^2)dA$, where $D=\{(x,y)\in\mathbb{R}^2; 1\le x^2+y^2\le 2, y\ge 0\}$

Exercise 7 *(20 points)* Let V be the volume of the region D that is bounded form below by the plane $z=0$, from above by the sphere $x^2+y^2+z^2=4$ and on the sides by the cylinder $x^2+y^2=1$

a. express V as an iterated triple integral cartesian coordinates in the order $dzdydx$

b. express V as an iterated triple integral cartesian coordinates in the order $dydzdx$

c. express V as an iterated triple integral spherical coordinates

d. express V as an iterated triple integral cylindrical coordinates, then evaluate the resulting integral.

Exercise 8 *(10 points)* Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2}\int_1^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2\,dzdrd\theta$$

(do not evaluate the integral)

good luck

(A) (1)

Math 201–Final Exam (Fall 04) B. Shayya

- Please write your **section number** on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)
(24 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(5,0,9)}^{(1,\pi,0)} (2x\cos y + yz)\,dx + (xz - x^2\sin y)\,dy + (xy)\,dz$$

Problem 2 (answer on pages **3** and **4** of the booklet.)
(24 pts) Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 3 (answer on pages **5** and **6** of the booklet.)
Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$.
(i) (8 pts) Set up the triple integrals in cylindrical coordinates that give the volume of D using the order of integration $dz\,r dr\,d\theta$. Then find the volume of D.
(ii) (6 pts) Set up the limits of integration for evaluating the integral of a function $f(x, y, z)$ over D as an iterated triple integral in the order $dy\,dz\,dx$.
(iii) (12 pts) Set up the triple integrals in spherical coordinates that give the volume of D using the order of integration $d\phi\,d\rho\,d\theta$.

Problem 4 (answer on pages **7** and **8** of the booklet.)
(25 pts) Integrate $g(x, y, z) = z$ over the surface of the prism cut from the first octant by the planes $z = x$, $z = 2 - x$, and $y = 2$.

Problem 5 (answer on pages **9**, **10**, and **11** of the booklet.)
Let S be the cone $z = 1 - \sqrt{x^2 + y^2}$, $0 \le z \le 1$, and let C be its base (i.e. C is the unit circle in the xy-plane). Find the counterclockwise circulation of the field

$$F(x, y, z) = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

around C
(a) (12 pts) directly,
(b) (8 pts) using Green's theorem, and
(c) (14 pts) using Stokes' theorem (i.e. by evaluating the flux of curl F outward through S).

Problem 6 (answer on pages **12** and **13** of the booklet.)
(25 pts) Let R be the region in the xy-plane bounded by the lines $y = 0$, $y = x$, $x + y = 4$, and $x + y = 9$. Use the transformation

$$x = uv, \qquad y = (1 - u)v$$

to rewrite

$$\int\int_R \frac{1}{\sqrt{x+y}}\,dx\,dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 7 (answer on page **14** of the booklet.)
(6 pts each) Determine which of the following series converge, and which diverge.

$$(a)\quad \sum_{n=1}^{\infty} \sqrt{n}\,\ln\left(1+\frac{1}{n^{2.1}}\right) \qquad (b)\quad \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{1.2}} \qquad (c)\quad \sum_{n=1}^{\infty} n(\sqrt[n]{n}-1)$$

Problem 8 (answer on pages **15** and **16** of the booklet.)
(i) (6 pts) Use Taylor's theorem to prove that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty < x < \infty).$$

(ii) (6 pts) Approximate

$$\int_0^{0.1} e^{-x^2}\,dx$$

with an error of magnitude less than 10^{-5}.

(iii) (6 pts) Show that

$$\int_0^{\infty} e^{-\pi x^2}\,dx = \frac{1}{2}.$$

(Hint. If $I = \int_0^{\infty} e^{-\pi x^2}dx$, then $I^2 = \int_0^{\infty}\int_0^{\infty} e^{-\pi(x^2+y^2)}dxdy$.)

(iv) (6 pts) Let E be the error resulting from the approximation

$$\int_0^{100} e^{-\pi x^2}\,dx \approx \frac{1}{2}.$$

Show that

$$|E| < \frac{e^{-5000\pi}}{2}.$$

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2007-2008

Final Exam

Name: .... ID #: ....

Exercise 1 *(12 points)* Discuss the convergence of the following series:

$$a)\ \sum_{n=0}^{+\infty} \frac{1}{\sqrt{n!}} \qquad b)\ \sum_{n=0}^{+\infty} \frac{n+10}{n\ln^3 n} \qquad c)\ \sum_{n=0}^{+\infty} (-1)^{n^2} \frac{1}{n^2+\sqrt{n}}$$

Exercise 2 *(10 points)* If $w = f(x, y)$ is differentiable and $x = r + s, y = r - s$, show that

$$\frac{\partial w}{\partial r} \times \frac{\partial w}{\partial s} = \left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2$$

Exercise 3 *(15 points)* Find the absolute extrema of $f(x, y) = 5 + 4x - 2x^2 + 3y - y^2$ on the region R bounded by the lines $y = 2, y = x$, and $y = -x$.

Exercise 4 *(8 points)* Use Lagrange multipliers to find the maximum and minimum of $f(x, y) = 4xy$ subject to $x^2 + y^2 = 8$.

Exercise 5 *(15 points)* Evaluate the integral $\displaystyle\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin(2z)}{4-z}\, dydzdx$

Exercise 6 *(10 points)*
Evaluate the integral $\displaystyle\int\int_D \cos(x^2 + y^2) dA$, where $D = \{(x, y) \in \mathbb{R}^2; 1 \leq x^2 + y^2 \leq 2, y \geq 0\}$

Exercise 7 *(20 points)* Let V be the volume of the region D that is bounded form below by the plane $z = 0$, from above by the sphere $x^2 + y^2 + z^2 = 4$ and on the sides by the cylinder $x^2 + y^2 = 1$

a. express V as an iterated triple integral cartesian coordinates in the order $dzdydx$

b. express V as an iterated triple integral cartesian coordinates in the order $dydzdx$

c. express V as an iterated triple integral spherical coordinates

d. express V as an iterated triple integral cylindrical coordinates, then evaluate the resulting integral.

Exercise 8 *(10 points)* Set up an integral in rectangular coordinates equivalent to the integral

$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2 dzdrd\theta$$

(do not evaluate the integral)

good luck

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Spring 2005

Final Exam

Name: .... ID #: ....

Exercise 1 *[9 points]* Discuss whether the following series converges or diverges:

a) $\displaystyle\sum_{n=1}^{+\infty} \frac{\sqrt[n]{n}}{1+n^2}$

b) $\displaystyle\sum_{n=1}^{+\infty} \frac{1.3.5.7...(2n-1)}{(2n)!}$

c) $\displaystyle\sum_{n=0}^{+\infty} \frac{e^n}{3^{n(n+1)}}$

Exercise 2 *[10 points]* Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n^2+1}} (x-\frac{1}{2})^n$

(be sure to check at the end points)

Exercise 3 a) *[5 points]* Find an equation of the tangent plane and a parametric equation for the normal line to the surface $z = 3x^2y - 5x + 1$ at the point $(2, 1, 3)$.

b) *[6 points]* Suppose that $z = e^{xy}$ and that $x = 2u + v, y = u/v$. Use the chain rule to find $\partial z/\partial u$ and $\partial z/\partial v$.

(give the expressions in terms in u and v)

c) *[5 points]* Prove or disprove: $g(x,y) = \dfrac{x^2y^2}{x^4+3y^4}$ can be extended by continuity at $(0,0)$. Justify.

Exercise 4 *[15 points]* Find the absolute minimum and maximum values of

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the closed triangular region R with vertices $(0,0)$, $(3,0)$, and $(0,5)$.

Exercise 5 *[10 points]* Evaluate the double integral

$$\int_0^2 \int_0^{\sqrt{1-(y-1)^2}} \frac{x}{x^2+y^2}\, dx dy$$

by converting it to a double integral in polar coordinates.

Turn over →

Exercise 6 *[10 points]* Let V be the volume of the tetrahedron bounded by the coordinates planes and the plane $z = 4-4x-2y$. Express V as iterated triple integral in cartesian coordinates in the order:

a) $dxdzdy$ *(do not evaluate the resulting integral).*

b) $dzdydx$, then evaluate the resulting integral.

Exercise 7 *[15 points]* Let V be the volume of the region R cut from the cone $z = \sqrt{x^2+y^2}$ by the two planes $z = 1$ and $z = 2$.

a) Sketch the region of integration.

b) Express V as iterated triple integral in cylindrical coordinates *(do not evaluate the resulting integral).*

c) Express V as iterated triple integral in spherical coordinates, then evaluate the resulting integral.

Exercise 8 *[15 points]*

a) Find the work done by the force field defined by

$$F(x,y,z) = 4y\mathbf{i} + 2xz\mathbf{j} + 3y\mathbf{k}$$

acting on an abject as it moves along the line segment from $(1,1,1)$ to $(2,2,1)$.

b) Evaluate the line integral $\oint_{(3,-2,0)}^{(1,0,\pi)} (2x\cos z - x^2)dx + (z-2y)dy + (y - x^2\sin z)dz$.

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2008-2009

Final Exam

Exercise 1 a. *(5 points)* If $f(u,v,w)$ is a differentiable function and if $u = x - y$, $v = y - z$, and $w = z - x$, show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

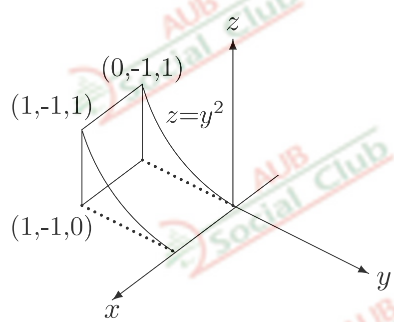
b. *(10 points)* Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 3x - y + 6$ on the circle $x^2 + y^2 = 4$

Exercise 2 *(10 points)* Convert to polar coordinates, then evaluate the following integral

$$\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2 dxdy$$

Exercise 3 *(12 points)* Here is the region of integration of the integral

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\ dy\ dx$$



Rewrite the integral as an equivalent iterated integral in the other 5 orders, then evaluate one of them

Exercise 4 Let V be the volume of the region D that is bounded by the paraboloid $z = x^2 + y^2$, and the plane $z = 2y$.

a) *(8 points)* Express V as an iterated triple integral in cartesian coordinates in the order $dz\ dx\ dy$ *(do not evaluate the integral).*

b) *(10 points)* Express V as an iterated triple integral in cylindrical coordinates, then evaluate the resulting integral.

(you may use the result: $\int \sin^4 x\ dx = -\frac{\sin^3 x \cos x}{4} - \frac{3\cos x \sin x}{8} + \frac{3x}{8}$*)*

Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$.

a) *(7 points)* Express V as an iterated triple integral in spherical coordinates in the order $d\rho\ d\phi\ d\theta$ *(do not evaluate the integral).*

b) *(8 points)* Express V as an iterated triple integral in spherical coordinates in the order $d\phi\ d\rho\ d\theta$ *(do not evaluate the integral).*

⟶

Exercise 6 a. *(6 points)* Find the work done by the force $F = x\mathbf{i} + y^2\mathbf{j} + (y - z)\mathbf{k}$ along the straight line from $(0,0,0)$ to $(1,1,1)$.

b. *(8 points)* Evaluate

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz$$

c. Find the *outward flux* of the field $F = (y - 2x)\mathbf{i} + (x + y)\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $y = x$ and $x + y = 1$.

i. *(10 points)* by direct calculation

ii. *(6 points)* by Green's theorem

good luck

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2009-2010

Final Exam

Exercise 1 **a.** *(3 points)* Find the directional derivative of $f(x, y) = x^2e^{-2y}$ at $P(1, 0)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

b. *(3 points)* The equation $1 - x - y^2 - \sin(xy)$ defines y as a differentiable function of x. Find dy/dx at the point $P(0, 1)$.

c. *(5 points)* Find the points on the surface $xy + yz + zx - x - z^2 = 0$, where the tangent plane is parallel to the xy-plane.

Exercise 2 *(12 points)* Find the absolute minimum and maximum values of $f(x, y) = x^2+xy+y^2 - 3x + 3y$ on the triangular region R cut from the first quadrant by the line $x + y = 4$.

Exercise 3 *(6 points)* Use Lagrange Multipliers to find the maximum and the minimum values of $f(x, y) = xy$ subject to the constraint $x^2 + y^2 = 10$.

Exercise 4 *(10 points)* Reverse the order of integration, then evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx$$

Exercise 5 *(10 points)* Let V be the volume of the region D that is bounded below by the xy-plane, above by the paraboloid $z = 9 - x^2 - y^2$, and lying outside the cylinder $x^2 + y^2 = 1$.

a. express V as an iterated triple integral in cylindrical coordinates, then *evaluate* the resulting integral.

b. express, but *do not evaluate* , V as an iterated triple integral in cylindrical coordinates in the order $drdzd\theta$.

Exercise 6 *(10 points)* Let V be the volume of the region that is bounded form below by the sphere $x^2 + y^2 + (z - 1)^2 = 1$ and from above by the cone $z = \sqrt{x^2 + y^2}$. Express V as an iterated triple integral in spherical coordinates, then evaluate the resulting integral *(sketch the region of integration)*.

Exercise 7 *(12 points)* Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$.
(do not evaluate any of the integrals)

⟶

Exercise 8 **a.** *(6 points)* Find the line integral of $f(x,y) = (x+y^2)/\sqrt{1+x^2}$ along the curve $C : y = x^2/2$ from $(1, 1/2)$ to $(0,0)$.

b. *(8 points)* Show that the differential form $2\cos y dx + (\frac{1}{y} - 2x\sin y)dy + (1/z)dz$ is exact, then evaluate

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y dx + \left(\frac{1}{y} - 2x\sin y\right) dy + (1/z)dz$$

c. Find the *counterclockwise circulation* of the field $F = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $x = 1$ and the curve $y = x^3$.

i. *(8 points)* by direct calculation

ii. *(8 points)* by Green's theorem

good luck

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2009-2010

Final Exam - solution

Exercise 1 **a.** Find the directional derivative of $f(x,y) = x^2e^{-2y}$ at $P(1,0)$ in the direction of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$

b. The equation $1 - x - y^2 - \sin(xy)$ defines y as a differentiable function of x. Find dy/dx at the point $P(0,1)$.

c. Find the points on the surface $xy + yz + zx - x - z^2 = 0$, where the tangent plane is parallel to the xy-plane.

Exercise 2 Find the absolute minimum and maximum values of $f(x,y) = x^2+xy+y^2-3x+3y$ on the triangular region R cut from the first quadrant by the line $x + y = 4$.

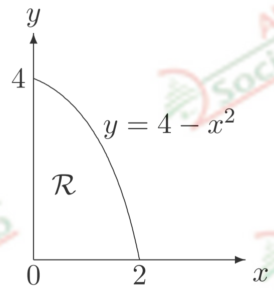
Exercise 3 Use Lagrange Multipliers to find the maximum and the minimum values of $f(x,y) = xy$ subject to the constraint $x^2 + y^2 = 10$.

Exercise 4 Reverse the order of integration, then evaluate the integral

$$I = \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dydx$$

<u>*solution*</u>: to express the integral in the order $dxdy$, we sketch the region of integration $\mathcal{R}$ in the xy-plane.

$$\begin{aligned} I &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy \\ &= \int_0^4 \frac{e^{2y}}{4-y}\left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} dy \\ &= \int_0^4 \frac{e^{2y}}{2}\, dy = \left[\frac{e^{2y}}{8}\right]_0^4 = \frac{e^8-1}{4} \end{aligned}$$



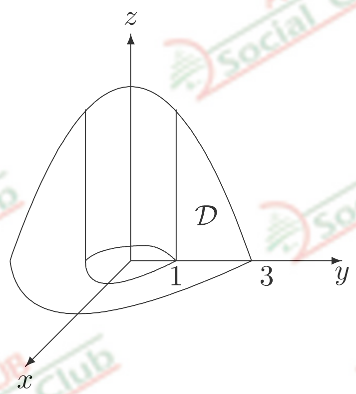
Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the paraboloid $z = 9 - x^2 - y^2$, and lying outside the cylinder $x^2 + y^2 = 1$.

<u>solution</u>:

a) $$V = \int_0^{2\pi} \int_1^3 \int_0^{9-r^2} r \; dzdrd\theta$$

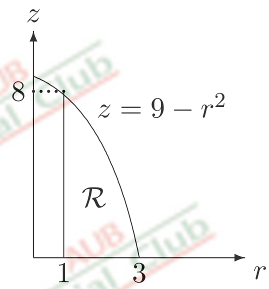
$$= 2\pi \int_1^3 r(9 - r^2)\, dr$$

$$= 2\pi \left[\tfrac{9}{2} r^2 - \tfrac{r^4}{4}\right]_1^3 = 32\pi$$



b) to express the integral in the order $drdzd\theta$, we sketch the region of integration $\mathcal{R}$ in the rz-plane.

$$V = \int_0^{2\pi} \int_0^{8} \int_1^{\sqrt{9-z}} r \; dr\, dz\, d\theta$$



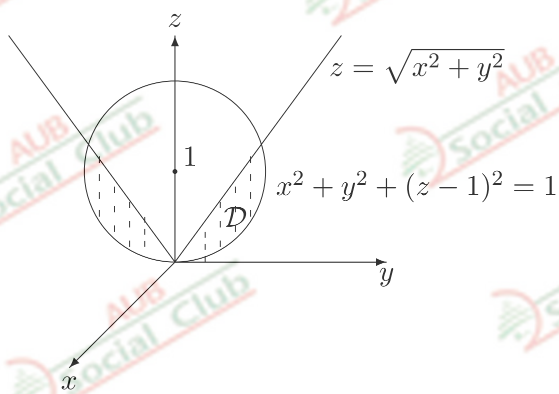
Exercise 6 Let V be the volume of the region that is bounded form below by the sphere $x^2 + y^2 + (z-1)^2 = 1$ and from above by the cone $z = \sqrt{x^2 + y^2}$. Express V as an iterated triple integral in spherical coordinates, then evaluate the resulting integral *(sketch the region of integration)*.

<u>solution</u>: The equation of the sphere $x^2 + y^2 + (z-1)^2 = 1$ in spherical coordinates is $\rho = 2\cos\phi$

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{2\cos\phi} \rho^2 \sin\phi \; d\rho \; d\phi \; d\theta$$

$$= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \frac{8}{3} \cos^3\phi \sin\phi \; d\phi \; d\theta$$

$$= \int_0^{2\pi} \frac{2}{3} \Big[- \cos^4\phi \Big]_{\pi/4}^{\pi/2} d\theta = \int_0^{2\pi} \frac{1}{6} \; d\theta = \frac{\pi}{3}$$



Exercise 7 Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$.
(do not evaluate any of the integrals)

Exercise 8 a. Find the line integral of $f(x,y) = (x+y^2)/\sqrt{1+x^2}$ along the curve $C: y = x^2/2$ from $(1, 1/2)$ to $(0,0)$.

<u>*solution*</u>: - $r(t) = (1-t)\mathbf{i} + \frac{(1-t)^2}{2}\mathbf{j}$, $0 \le t \le 1$;

- $v(t) = \frac{dr}{dt} = -\mathbf{i} - (1-t)\mathbf{j}$, and $|v(t)| = \sqrt{1+(1-t)^2}$;

- $f(t) = \frac{(1-t)+\frac{(1-t)^4}{4}}{\sqrt{1+(1-t)^2}}$, hence

$$\int_C f(s)ds = \int_0^1 f(t).|v(t)|dt = \int_0^1 \left[(1-t) + \frac{(1-t)^4}{4}\right] dt = 11/20$$

b. Show that the differential form $2\cos y dx + (\frac{1}{y} - 2x\sin y)dy + (1/z)dz$ is exact, then evaluate

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y dx + \left(\frac{1}{y} - 2x\sin y\right) dy + (1/z)dz$$

<u>*solution*</u>: $f(x,y,z) = 2x\cos y + \ln(yz) + C$ is a potential function (check it!), hence

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y dx + \left(\frac{1}{y} - 2x\sin y\right) dy + (1/z)dz = [2x\cos y + \ln(yz) + C]_{(0,2,1)}^{(1,\pi/2,2)} = \ln(\pi/2)$$

c. Find the *counterclockwise circulation* of the field $F = 2xy^3\mathbf{i} + 4x^2y^2\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y=0$, $x=1$ and the curve $y = x^3$.

i) direct calculation: *cicculation* $= \oint_C Mdx + Ndy$

$C_1 : r_1(t) = t\mathbf{i}$, $0 \le t \le 1$; $Mdx + Ndy = 0$, and $\int_{C_1} dx + Ndy = 0$

$C_2 : r_2(t) = \mathbf{i} + t\mathbf{j}$, $0 \le t \le 1$; $Mdx + Ndy = 4t^2dt$, and $\int_{C_2} Mdx + Ndy = \int_0^1 4t^2dt = 4/3$

$C_3 : r_3(t) = (1-t)\mathbf{i} + (1-t)^3\mathbf{j}$, $0 \le t \le 1$; $Mdx + Ndy = -14(1-t)^{10}dt$, and

$$\int_{C_3} Mdx + Ndy = \int_0^1 -14(1-t)^{10}dt = -14/11$$

$$circulation(F) = \oint_C Mdx + Ndy = 0 + 4/3 - 14/11 = 2/33$$

ii) Green's theorem: $circulation(F) = \int\int_R (\mathbf{curlF}) \cdot \mathbf{k}\, dA$

$$(\mathbf{curlF}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 8xy^2 - 6xy^2 = 2xy^2$$

$$\int\int_R (\mathbf{curlF}) \cdot \mathbf{k}\, dA = \int_0^1 \int_0^{x^3} 2xy^2\, dy\, dx = \int_0^1 2x \left[\frac{y^3}{3}\right]_0^{x^3} dx = 2/3 \int_0^1 x^{10}dx = 2/33$$

American University of Beirut
MATH 201
Calculus and Analytic Geometry III
Fall 2008-2009

Final Exam - solution

Exercise 1 a. If $f(u,v,w)$ is a differentiable function and if $u = x-y$, $v = y-z$, and $w = z-x$, show that $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\cdot\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$; $\frac{\partial f}{\partial y} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$, and $\frac{\partial f}{\partial z} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$, and hence $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0$

b. Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x,y) = 3x - y + 6$ on the circle $x^2 + y^2 = 4$

$$\nabla f = \lambda \nabla g \Leftrightarrow \begin{cases} 3 = 2\lambda x \\ -1 = 2\lambda y \\ x^2 + y^2 = 4 \quad (g(x,y) = 0) \end{cases} \Leftrightarrow \begin{cases} \frac{3}{2\lambda} = x \\ -\frac{1}{2\lambda} = y \\ x^2 + y^2 = 4 \end{cases}$$

$x^2 + y^2 = 4 \Rightarrow \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} = 4 \Rightarrow \lambda^2 = \frac{10}{16} \Rightarrow \lambda = \pm\frac{\sqrt{10}}{4}$

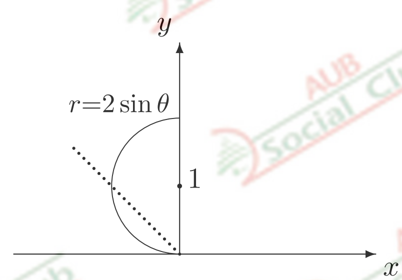
for $\lambda = \frac{\sqrt{10}}{4}$; $x = \frac{6}{\sqrt{10}}$, and $y = -\frac{2}{\sqrt{10}}$, and $f(x,y) = 2\sqrt{10} + 6$ (maximum value of f)

for $\lambda = -\frac{\sqrt{10}}{4}$; $x = -\frac{6}{\sqrt{10}}$, and $y = +\frac{2}{\sqrt{10}}$, and $f(x,y) = -2\sqrt{10} + 6$ (minimum value of f)

Exercise 2 Convert to polar coordinates, then evaluate the following integral

$$\int_0^2 \int_{-\sqrt{1-(y-1)^2}}^0 xy^2\,dx\,dy = \int_{\pi/2}^{\pi} \int_0^{2\sin\theta} r^4 \cos\theta \sin^2\theta \; dr\,d\theta$$

$$= \int_{\pi/2}^{\pi} \frac{32}{5} \cos\theta \sin^7\theta \; d\theta = \frac{4}{5}\left[\sin^8\theta\right]_{\pi/2}^{\pi} = -\frac{4}{5}$$

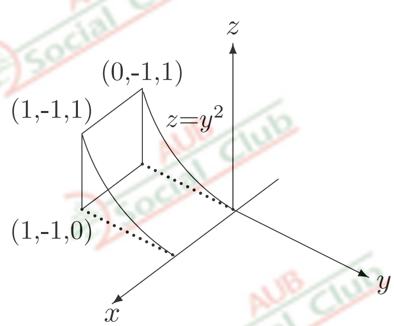


Exercise 3 $\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\;dy\;dx = 1/3$

$$\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\;dx\;dy = \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\;dz\;dx$$

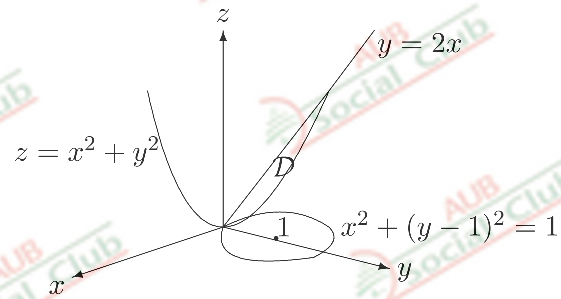
$$= \int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\;dx\;dz = \int_{-1}^0 \int_0^{y^2} \int_0^1 dx\;dz\;dy$$

$$= \int_0^1 \int_{-1}^{-\sqrt{z}} \int_0^1 dx\;dy\;dz$$



Exercise 4 Let V be the volume of the region D that is bounded by the paraboloid $z = x^2 + y^2$, and the plane $z = 2y$.

a) cartesian coordinates:



$$V = \int_0^2 \int_{-\sqrt{1-(y-1)^2}}^{\sqrt{1-(y-1)^2}} \int_{x^2+y^2}^{2y} dz\, dx\, dy$$

b) cylindrical coordinates:

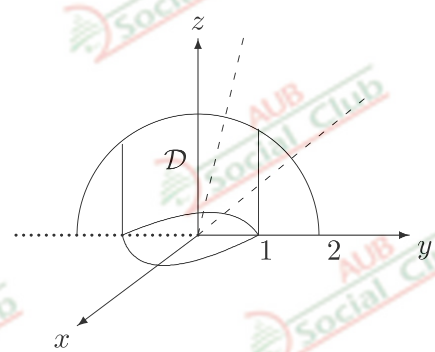
$$V = \int_0^{\pi} \int_0^{2\sin\theta} \int_{r^2}^{2r\sin\theta} r\, dz\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^{2\sin\theta} (2r^2 \sin\theta - r^3)\, dr\, d\theta = \int_0^{\pi} \left[\frac{2}{3} r^3 \sin\theta - \frac{r^4}{4} \right]_{r=0}^{2\sin\theta} d\theta = \int_0^{\pi} \frac{4}{3} \sin^4\theta\, d\theta$$

$$= \tfrac{4}{3} . \tfrac{3\pi}{8} = \tfrac{\pi}{2}$$

Exercise 5 Let V be the volume of the region D that is bounded below by the xy-plane, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$.

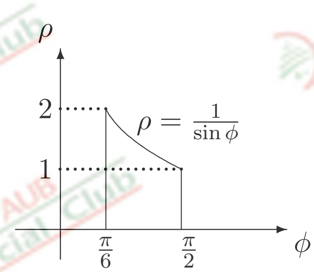
a) spherical coordinates: order $d\rho\, d\phi\, d\theta$



$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^2 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta +$$

$$\int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

b) spherical coordinates: order $d\phi\, d\rho\, d\theta$



$$V = \int_0^{2\pi} \int_0^2 \int_0^{\frac{\pi}{6}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta +$$

$$\int_0^{2\pi} \int_0^1 \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta +$$

$$\int_0^{2\pi} \int_1^2 \int_{\frac{\pi}{6}}^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta$$

the answer of part b) can also be written:

$$\int_0^{2\pi} \int_0^1 \int_0^{\frac{\pi}{2}} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta + \int_0^{2\pi} \int_1^2 \int_0^{\sin^{-1}(1/\rho)} \rho^2 \sin\phi\, d\phi\, d\rho\, d\theta \quad \text{(why??)}$$

Exercise 6 a. Find the work done by the force $F = x\mathbf{i} + y^2\mathbf{j} + (y-z)\mathbf{k}$ along the straight line from $(0,0,0)$ to $(1,1,1)$.

$r(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$; $\frac{dr}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $F(t) = t\mathbf{i} + t^2\mathbf{k}$, and $F \centerdot \frac{dr}{dt} = t + t^2$, hence

$$W = \int_0^1 (t+t^2)dt = 5/6$$

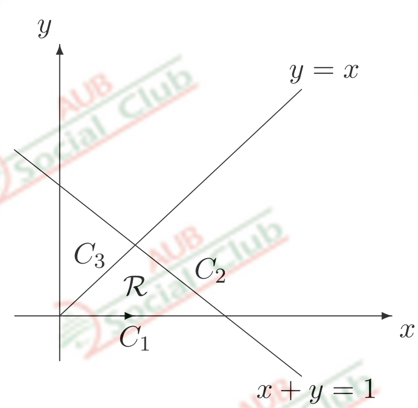
b. Evaluate

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz$$

$f(x,y,z) = x\ln z + e^x \sin y - \frac{z^2}{2} + C$ is a potential function (check it!), hence

$$\int_{(0,0,1)}^{(1,\pi/2,e)} (\ln z + e^x \sin y)dx + e^x \cos y dy + (x/z - z)dz = \left[x\ln z + e^x \sin y - \frac{z^2}{2} \right]_{(0,0,1)}^{(1,\pi/2,e)} = \frac{3}{2} + e - \frac{e^2}{2}$$

c. Find the *outward flux* of the field $F = (y-2x)\mathbf{i} + (x+y)\mathbf{j}$ across the curve C in the first quadrant, bounded by the lines $y = 0$, $y = x$ and $x + y = 1$.



i) direct calculation: $Flux = \oint_C Mdy - Ndx$

$C_1 : r_1(t) = t\mathbf{i}$, $0 \le t \le 1$; $Mdy - Ndx = -tdt$

and $\int_{C_1} Mdy - Ndx = \int_0^1 -tdt = -1/2$

$C_2 : r_2(t) = (1-t)\mathbf{i} + t\mathbf{j}$, $0 \le t \le 1/2$;

$Mdy - Ndx = (1-3t)dt$

and $\int_{C_2} Mdy - Ndx = \int_0^{1/2} (1-3t)dt = -1/8$

$C_3 : r_3(t) = (\frac{1}{2} - t)\mathbf{i} + (\frac{1}{2} - t)\mathbf{j}$, $0 \le t \le 1/2$; $Mdy - Ndx = (\frac{3}{2} - 3t)dt$, and

$$\int_{C_3} Mdy - Ndx = \int_0^{1/2} (\frac{3}{2} - 3t)dt = 3/8$$

$$Flux(F) = \oint_C Mdy - Ndx = -1/2 - 1/8 + 3/8 = -1/4$$

ii) Green's theorem: $Flux = \int\int_R \mathbf{divF}dA$

$$divF = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = -2 + 1 = -1$$

$$\int\int_R \mathbf{divF}dA = \int_0^{1/2} \int_y^{1-y} - dxdy = \int_0^{1/2} (2y-1)dy = -1/4$$

Final Exam-Math 201

- Write your name and your I.D. on the booklet
- The duration of the test is two hours
- Calculators are allowed

1. (5 points) Determine whether $\sum_{n=1}^{\infty} \frac{(tan^{-1}n)^2}{n^2+1}$ converge or diverge. Please provide all the details.

2. (10 points) a) Show that for $\mid x \mid < 1$, we have

$$ln\frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + ...\right).$$

   (Hint: Determine the expansion of $ln(1+x)$ and $ln(1-x)$ first to determine the expansion of $ln\frac{1+x}{1-x}$).
   b) How many terms of the Mclaurin series for $ln(1+x)$ should you add to be sure of calculating $ln1.1$ with an error of magnitude less than 10^{-8}.

3. (10 points) Find the maximum and the minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ by using the method of Lagrange multipliers.

4. (10 points) Your company manufactures right circular cylindrical molasses storage tanks that are 25 feet high with a radius of 5 feet. How sensitive are the tanks' volumes to small variations in height and radius. After finding the formula for the change in volume, calculate the change in volume if the radius changes from 5 to 5.1 and the height changes from 25 to 24.3.(Hint: The volume of the tank is given by $V = \pi r^2 h$).

5. (5 points) Find all the local extreme values of the function $f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$.

6. (10 points)Suppose that $f(x, y, z)$is given by

$$f(x, y, z) = x^3 - xy^2 - z$$

a) Find the rate of change of f at $P(1, 1, 0)$ in the direction of the vector $v = 2i - 3j + 6k$.
b) In what direction does f change most rapidly at P.
c) Use the linear approximation to obtain an approximate value of f(1.01,1.01,0.1).

7. (10 points)Sketch the regions of integration and evaluate the integrals

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y} dydx$$

and

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2dydx}{(1 + x^2 + y^2)^2}$$

8. (5 points) Find the volume of the region in the first octant bounded by the coordinate planes and the planes $x + z = 1$ and $y + 2z = 2$.

9. (5 points) Evaluate $\int\int\int_E \sqrt{x^2 + z^2} dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.

10. (5 points) Set up but **do not evaluate** the integral for the volume of the solid that is bounded below by the xy-plane, on the sides by the sphere $x^2 + y^2 + z^2 = 2$ and above by the cone $z = \sqrt{x^2 + y^2}$.

11. (10 points) Given the vector field $F = (e^x cosy + yz)i + (xz - e^x siny)j + (xy + z)k$.
a) Show that F is conservative and find a potential function.
b) Find the work done by the field F along the parabolic path joining the point $(-1, 3, 9)$ to the point $(1, 6, -4)$.

12. (15 points) Given a vector field $F = yi - xj$ and the closed curve C given by parts of the parabola $y = x^2$ and the line $y = 2$. C has counterclockwise orientation and please use the outward normal.
a) Directly evaluate the circulation integral $\int_C F.Tds$,by parametrization.
b) Use Green's theorem to reevaluate the circulation.
c) Use Green's theorem to calculate the flux integral $\int_C F.nds$.

Quiz 1

- Please write your section number and your name on the booklet
- Answer each problem on a seperate page of the booklet.

1. (10 points) Given a telescoping series

$$S = \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right).$$

   Let s_N denote the Nth partial sum of S. First find a simple formula for s_N in terms of N. Then determine how much should N be to guarantee that $\mid S - s_N \mid \leq 0.01$.

2. (20 points) Determine whether the following series converge or diverge. Be sure to indicate what test you are using and carry out all work related to that test.

   - $\sum_{n=1}^{\infty} \frac{2+cosn}{n^2}$.
   - $\sum_{n=1}^{\infty} \frac{n-1}{\sqrt{n^3-1}}$.
   - $\sum_{n=1}^{\infty} \left(\frac{5n+lnn}{8n-lnn}\right)^n$.

- $\sum_{n=1}^{\infty} \frac{n+3}{3^n}$.

3. (20 points) Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{lnn}$ converges absolutely, converges conditionally, or diverges. Be sure to indicate what test you are using and carry out all work related to that test. Also estimate the error in using the 5th partial sum to approximate the total sum.

4. - (10 points) Find the radius and interval of convergence of the power series
$$\sum_{n=1}^{\infty} \frac{n(x-1)^n}{4^n}.$$
   - (10 points) Find only the radius of convergence of the power series
$$\sum_{n=0}^{\infty} \frac{n!(n+2)!}{(2n)!} x^n$$
.

5. (20 points) Given that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x. Find the power series in powers of x for $f(x) = \int e^x dx$ and determine its radius of convergence. Do you get the series of e^x, explain.

6. (10 points)
   - Show that $tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$ for $| x |< 1$.
   - Find also the Taylor polynomial $p_1(x)$ generated by $f(x) = tan^{-1}x$ at $x = 0$. Then use the alternating series estimation theorem to estimate the error resulting from the approximation $tan^{-1}(-0.1) \sim p_1(-0.1)$.

   Hint: The derivative of $f(x) = tan^{-1}x$ is $\frac{1}{1+x^2}$

Math 201–Exam 1 (Fall 06) B. Shayya

- Please write your **section number** on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)
(9 pts each) Which of the following series converge, and which diverge? When possible, find the sum of the series.

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-3)^{n+1}}{5^n}+\frac{2^{n-1}}{3^{n+2}}\right)$ (ii) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{\sqrt{n}\,(\ln n)^{100}}$ (iii) $\displaystyle\sum_{n=1}^{\infty}\frac{\cos(2^n+n^2)}{e^n}$ (iv) $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}-1}{n^{0.2}}$

Problem 2 (answer on pages **3** and **4** of the booklet.)
(20 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{4^n\,n!\,n!}{(2n)!}(x-5)^n.$$

For what values of x does the series converge absolutely? Conditionally?

Problem 3 (answer on pages **5** and **6** of the booklet.)
(i) (8 pts) Prove that

$$\arctan x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1},\qquad |x|<1.$$

(ii) (4 pts) Let $f(x)=\arctan x$. Find $f^{(2007)}(0)$.
(iii) (7 pts) Find the Taylor polynomial $p_1(x)$ generated by $f(x)=\arctan x$ at the point $x=0$. Then use the alternating series estimation theorem to estimate the error resulting form the approximation

$$\arctan(-0.1)\approx p_1(-0.1).$$

Does $p_1(-0.1)$ tend to be too large or too small?
(iv) (6 pts) Use Taylor's theorem to estimate the error resulting from the approximation

$$\arctan(1/\sqrt{3})\approx p_1(1/\sqrt{3}).$$

Does $p_1(1/\sqrt{3})$ tend to be too large or too small? (Notice that $p_1(x)=p_2(x)$. You may need the fact that the third derivative of the function $f(x)=\arctan x$ is $f'''(x)=(6x^4+4x^2-2)/(1+x^2)^4$.)
(v) (4 pts) Find a power series expansion for the function

$$g(x)=\frac{\arctan x}{1-x}$$

about the point $x=0$. (It is enough to find the first four terms of the power series expansion.)

Problem 4 (answer on page **7** of the booklet.)
Consider the sequence

$$a_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}.$$

(i) (6 pts) Prove that $\ln(n+1)\le a_n\le 1+\ln n$ for all n.
(ii) (3 pts) Does $\lim_{n\to\infty}a_n$ exist? Why or why not?
(iii) (6 pts) What about $\lim_{n\to\infty}a_n/\ln n$?

Wed, March 8 - 2006
5:00 - 7:00 p.m

Math 201 – Exam 1 (Spring 06)

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)
(8 pts each) Which of the following sequences converge, and which diverge? Find the limit of each convergent sequence.

(i) $a_n = \dfrac{5^n + n^9 - 8n^2 - 3}{7n^9 + 2n + 5^{n+1}}$ (ii) $b_n = \dfrac{\sin(n! + 2n^3 - 7)}{\sqrt{n+3} - 1}$ (iii) $c_n = (3n+2)\arctan\left(\dfrac{1}{n}\right)$

Problem 2 (answer on pages 2 and 3 of the booklet.)
(9 pts each) Which of the following series converge, and which diverge? When possible, find the sum of the series.

(i) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{2^n} + \frac{3^{n-1}}{5^{n+2}}\right)$ (ii) $\displaystyle\sum_{n=3}^{\infty}\frac{1}{n(n+1)}$ (iii) $\displaystyle\sum_{n=2}^{\infty}\frac{\cos n}{n^n}$ (iv) $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^{1.3}}$

Problem 3 (answer on pages 4 and 5 of the booklet.)
(20 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}(e^{1/n} - 1)(x-4)^n.$$

For what values of x does the series converge absolutely? Conditionally?

Problem 4 (answer on page 6 of the booklet.)
(i) (5 pts) Find a power series expansion for $f(x) = \ln(1+x)$ about the point $x = 0$. Also find the Taylor polynomials $p_1(x)$ and $p_2(x)$ generated by f at the point $x = 0$.

(ii) (5 pts) Use the alternating series estimation theoren to estimate $\ln(1.1)$ with an error of magnitude less than 10^{-4}. Does your estimate tend to be too large or too small?

(iii) (5 pts) Use Taylor's theorem to show that

$$|f(x) - p_2(x)| \le \frac{8}{3}|x|^3, \quad \text{for} \quad |x| \le \frac{1}{2}.$$

(iv) (5 pts) Find

$$\lim_{n\to\infty} e^{4n}\left(1 - \frac{4}{n}\right)^{n^2}.$$

Math 201 – Exam 1 (Spring 07) B. Shayya

- Please write your **section number** on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)
(9 pts each) Which of the following series converge, and which diverge? When possible, find the sum of the series.

(i) $\sum_{n=1}^{\infty}\left(\frac{(-2)^{n-1}}{3^n}+\frac{1}{5^{n+2}}\right)$ (ii) $\sum_{n=1}^{\infty}\left(\frac{n-3}{n+1}\right)^n$ (iii) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ (iv) $\sum_{n=1}^{\infty}\ln(1+\frac{1}{n^3})$

Problem 2 (answer on page **3** of the booklet.)
(15 pts) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}(x-3)^n.$$

For what values of x does the series converge absolutely? Conditionally?

Problem 3 (answer on pages **4** and **5** of the booklet.)
(i) (10 pts) Find a power series expansion for $f(x)=\sqrt{1+x}$ about the point $x=0$. Also find the Taylor polynomials $p_1(x)$ and $p_2(x)$ generated by f at the point $x=0$.

(ii) (5 pts) Express $\int\sqrt{1+x^4}\,dx$ as a power series.

(iii) (5 pts) Approximate $\sqrt{1.01}$ by $p_2(?)$ and use the alternating series estimation theorem to estimate the resulting error.

(iv) (9 pts) Approximate $\sqrt{0.85}$ by $p_1(?)$ and use Taylor's theorem to estimate the resulting error.

Problem 4 (answer on page **6** of the booklet.)
(a) (10 pts) Find the Fourier series of the function $f(x)=x$, $0\le x\le 2\pi$.

(b) (5 pts) Find the sum of the series in part (a) for $0\le x\le 2\pi$.

(c) (5 pts) Use the result of part (b) to find

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}.$$

Math 201–Exam 2 (Fall 04) Prof. B. Shayya and Dr. H. Yamani

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)

(15 pts) Find the domain and range of the function $f(x,y,z) = 3/(x^2+y^2+z^2-9)$ and identify its level surfaces. Determine if the domain of f is an open region, a closed region, or neither. Also, decide if the domain is bounded or unbounded.

Problem 2 (answer on page 2 of the booklet.)

(a) (7 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x}\,y}{\sqrt{y^2+x^2}}$$

exist? Why or why not?

(b) (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x}\,y}{\sqrt{y^2-x^2}}?$$

Problem 3 (answer on pages 3, 4, and 5 of the booklet.)

Suppose that the derivative of the function $f(x,y,z)$ at the point $(1,1,1)$ is greatest in the direction of $A = 6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, and that in this direction the value of the derivative is $\sqrt{6}$. Also suppose that

$$f(3,0,-1) = 1, \quad \nabla f(3,0,-1) = 3\mathbf{i}-\mathbf{j}+5\mathbf{k}, \quad \nabla f(3,2,1) = 6\mathbf{i}-2\mathbf{j}+\mathbf{k} \quad \text{and} \quad \nabla f(0,-1,1) = \mathbf{i}+\mathbf{j}+\mathbf{k}.$$

(a) (5 pts) Find the derivative of f at the point $(3,2,1)$ in the direction of $\mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$.

(b) (10 pts) Find $\nabla f(1,1,1)$.

(c) (4 pts) Is there a unit vector u such that $D_u f(3,0,-1) = 6$? Give reasons for your answer.

(d) (6 pts) Find the line normal to the surface $f(x,y,z) = 1$ at the point $(3,0,-1)$.

(e) (10 pts) Let

$$x = \alpha, \qquad y = \beta - 2, \qquad z = \beta - \alpha, \qquad \text{and} \qquad \omega = f(x,y,z).$$

Find $\partial\omega/\partial\alpha$ and $\partial\omega/\partial\beta$ at the point $(\alpha,\beta) = (3,2)$.

(f) (10 pts) Let $\omega = \omega(\alpha,\beta)$ be as in part (e). Find a plane tangent to the surface

$$\omega(\alpha,\beta) = 2\gamma^2 - 1$$

in the $\alpha\beta\gamma$-space. (Hint. Start by finding a point $(\alpha,\beta,\gamma) = (?,?,?)$ on the surface.)

Problem 4 (answer on page 6 of the booklet.)

(a) (9 pts) Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < -1, \\ 1 & \text{if } -1 \le x \le 1, \\ 0 & \text{if } 1 < x < \pi. \end{cases}$$

(Notice that $L = \pi$, not 1.)

(b) (10 pts) Find the sum of the series in part (a) for $-\pi \le x \le \pi$.

(c) (6 pts) Use the result of part (b) to find

$$\sum_{n=1}^{\infty} \frac{\sin n}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{\sin(2n)}{n}.$$

Math 201 – Exam 2 (Spring 04) B. Shayya

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)

(a) (4 pts) Write the point $(x, y) = (\sqrt{2}, \sqrt{2})$ in polar coordinates.
(b) (8 pts) Find all polar coordinate representations of the point $(r, \theta) = (5, \pi/3)$.
(c) (6 pts) Graph the polar curve $r = 1 + 2\cos\theta$ for $0 \le \theta \le \pi$.

Problem 2 (answer on page 2 of the booklet.)

(16 pts) Find the tangent plane and normal line of the surface $z = x^2 + y^2$ at the point $(1, 2, 3)$.

Problem 3 (answer on page 3 of the booklet.)

(16 pts) Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = 2x^3 + 6x^2 - 4y^3 + 3y^2.$$

Problem 4 (answer on page 4 of the booklet.)

(a) (6 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^5}{x^6 + x^2y^4}$$

exist? Why or why not?
(b) (8 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^5}{3y - 2x}?$$

Problem 5 (answer on pages 5 and 6 of the booklet.)

Suppose that the derivative of the function $f(x, y, z)$ at the point $(1, 2, 1)$ is greatest in the direction of $A = \mathbf{i} - \mathbf{j} + \sqrt{2}\mathbf{k}$, and that in this direction the value of the derivative is 12. Also suppose that

$f(1, 1, 1) = 22$, $\nabla f(1, 1, 1) = \mathbf{i}+2\mathbf{j}-2\mathbf{k}$, $\nabla f(3, 5, -3) = 2\mathbf{i}-3\mathbf{j}+\mathbf{k}$ and $\nabla f(0, -1, 1) = \mathbf{i}+\mathbf{j}+\mathbf{k}$.

(a) (9 pts) Find $\nabla f(1, 2, 1)$.
(b) (3 pts) Is there a unit vector u such that $D_u f(3, 5, -3) = -4$? Give reasons for your answer.
(c) (12 pts) Approximate $f(1.1, 1.05, 0.95)$
(d) (6 pts) Let

$$x = r + s, \quad y = r + 2s, \quad z = r - s^2, \quad \text{and} \quad w = f(x, y, z).$$

Find $\partial w/\partial s$ at the point $(r, s) = (1, 2)$.
(e) (6 pts) Let $w = w(r, s)$ be as in part (d). Find a line tangent to the curve

$$w(r, s) = 22(r + s)$$

in the rs-plane. (Hint. Start by finding a point $(r, s) = (?, ?)$ on the curve.)

Math 201 – Exam 2 (Fall 06) B. Shayya

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page 1 of the booklet.)

(16 pts) Find the tangent plane and normal line of the surface $z = x + x^2 + y^2$ at the point $(-1, 2, 4)$.

Problem 2 (answer on page 2 of the booklet.)

(a) (11 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x^5y + x^2y^2}{x^4+y^2}$$

exist? Why or why not?

(b) (5 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^4}?$$

(c) (6 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^5y}{x^4+y}?$$

Problem 3 (answer on pages 3, 4 and 5 of the booklet.)

(a) (14 pts) Find the domain and range of the function

$$f(x,y) = \begin{cases} \sqrt{1-x^2-y^2} & \text{if } y \geq 0, \\ \frac{2}{\sqrt{1-4x^2-4y^2}} & \text{if } y < 0. \end{cases}$$

Determine if the domain of f is an open region, a closed region, or neither. Also, decide if the domain is bounded or unbounded.

(b) (6 pts) Find the level curves $f(x,y) = 3/4$ and $f(x,y) = 3$.

(c) (15 pts) Let

$$x = \frac{r+3s}{6+r+s}, \qquad y = \frac{r+\ln s}{1+r}, \quad \text{and} \quad w = f(x,y).$$

Find $\partial w/\partial r$ and $\partial w/\partial s$ at the point $(r,s) = (1,1)$. Also, find the derivative of w at the point $(r,s) = (1,1)$ in the direction of $\mathbf{i}+\mathbf{j}$.

(d) (7 pts) Let $w = w(r,s)$ be as in part (c). Find a line tangent to the curve

$$2\sqrt{2}\, w(r,s) = r^5 + s^3$$

in the rs-plane.

Problem 4 (answer on page 6 of the booklet.)

(a) (10 pts) Find the Fourier series of the function

$$f(x,y) = \begin{cases} x^2 & \text{if } 0 \leq x \leq \pi, \\ 0 & \text{if } \pi < x \leq 2\pi. \end{cases}$$

(See back page for relevant integrals.)

(b) (5 pts) Find the sum of the series in part (a) for $0 \leq x \leq 2\pi$.

(c) (5 pts) Use the result of part (b) to find $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Needed

$$\int x^2\cos(nx)\,dx = \frac{x^2\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}$$

$$\int x^2\sin(nx)\,dx = -\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}$$

Math 201 – Exam 2 (Spring 07) B. Shayya

- Please write your section number on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on page **1** of the booklet.)
(13 pts) Find the domain and range of the function

$$f(x,y,z) = \frac{2}{1 - x^2 - y^2 - z^2}.$$

Determine if the domain of f is an open region, a closed region, or neither. Also, decide if the domain is bounded or unbounded.

Problem 2 (answer on page **2** of the booklet.)
(a) (11 pts) Does

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^3 \sin e^x}{x^8 + y^2}$$

exist? Why or why not?
(b) (9 pts) What about

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y^3 \sin e^x}{x^8 - y^2}?$$

Problem 3 (answer on pages **3** and **4** of the booklet.)
Consider the function $f(x,y) = y + x^2 + y^2$.
(a) (12 pts) Find the tangent plane and normal line of the surface $z = f(x,y)$ at the point $(2,-1,4)$.
(b) (5 pts) Suppose the equation $f(x,y) - z^3 = \sin z$ defines z implicitly as a function of (x,y). Find $\partial z/\partial y$.
(c) (9 pts) Prove that $f(x,y)$ is differentiable everywhere in the plane.

Problem 4 (answer on pages **5** and **6** of the booklet.)
Suppose that the derivative of the function $f(x,y,z)$ at the point $(1,1,1)$ is greatest in the direction of $A = 6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$, and that in this direction the value of the derivative is $\sqrt{6}$. Also suppose that

$$f(3,0,0) = 1, \quad \nabla f(3,0,0) = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}, \quad f(3,2,1) = 3 \quad \text{and} \quad \nabla f(3,2,1) = 6\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

(a) (6 pts) Find the derivative of f at the point $(3,2,1)$ in the direction of $\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$.
(b) (10 pts) Find $\nabla f(1,1,1)$.
(c) (5 pts) Is there a unit vector u such that $D_u f(3,2,1) = -7$? Give reasons for your answer.
(d) (10 pts) Let

$$x = r^2 + s, \quad y = rs, \quad z = \frac{2r}{s}, \quad \text{and} \quad w = f(x,y,z).$$

Find $\partial w/\partial s$ at the point $(r,s) = (1,2)$.
(e) (10 pts) Let $w = w(r,s)$ be as in part (c). Find a plane tangent to the surface

$$\frac{2}{1-t} + 3 = w(r,s)$$

in the rst-space. (Hint. Start by finding a point (r_0, s_0, t_0) on the surface.)

$6(1) - 2(r) + 1\left(-\frac{2r}{s^2}\right)$

$1 - 2 - \frac{2}{4}$

- Please write your section number on your booklet.
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- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(a) (8 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(0,0,0)}^{(1,0,2)} (e^x \cos y + yz)\,dx + (xz - e^x \sin y)\,dy + (xy + z)\,dz$$

(b) (10 pts) Find the counterclockwise circulation and outward flux of the field $F(x,y) = (x + e^x \sin y)\mathbf{i} + (x + e^x \cos y)\mathbf{j}$ around and across the cardioid $r = 2(1+\cos\theta)$.

Problem 2 (answer on pages 4 and 5 of the booklet.)

(a) (8 pts) Find the volume of the region in the first octant that is bounded by the coordinate planes, the plane $2x + 3z - 12 = 0$, and the surface $y = \frac{1}{2}z^2$.

(b) (10 pts) Let D be the smaller region cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/6$. Evaluate $\int\int\int_D z\,dV$.

Problem 3 (answer on pages 6 and 7 of the booklet.)

(a) (8 pts) Find the points on the sphere $x^2 + y^2 + z^2 = 30$ where $f(x,y,z) = x - 2y + 5z$ has its maximum and minimum values.

(b) (10 pts) Suppose $f(x,y,z)$ is a differentiable function with $\nabla f(0,0,0) = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\nabla f(2,0,0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Let

$$x = 2\sqrt{1 - r^2 - s^2}, \qquad y = 3r, \qquad z = 5s, \qquad \text{and} \qquad w = f(x,y,z).$$

Find $\partial w/\partial r$ and $\partial w/\partial s$ at the point $(r,s) = (0,0)$.

Problem 4 (answer on page 8 of the booklet.)

(12 pts) Integrate $g(x,y,z) = x\sqrt{y^2+4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Problem 5 (answer on pages 9, 10, and 11 of the booklet.)

(a) (6 pts) Find the Fourier series of the function $f(x) = x$ on the interval $-\pi < x < \pi$.

(b) (6 pts) Use the series in part (a) to find $\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$.

(c) (6 pts) Determine whether the series $\sum_{n=1}^{\infty} (\ln n)\left(e^{1/n} - 1\right)^{13}$ converges or diverges.

Problem 6 (answer on pages 12 and 13 of the booklet.)

(16 pts) Let D be the disk $x^2 + y^2 \leq 3/4$. Use the transformation

$$x = u\sqrt{1 - \frac{u^2}{4} - \frac{v^2}{2}}, \qquad y = v\sqrt{1 - \frac{v^2}{4}}$$

to rewrite

$$\int\int_D 8xy\sqrt{1 - x^2 - y^2}\,dx\,dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G. (Hint: $\sqrt{1 - x^2 - y^2} = 1 - \frac{1}{2}(u^2 + v^2)$.)

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- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(a) (8 pts) Find the line integral of $f(x, y, z) = x + y + z$ over the straight-line segment from (1, to $(0, -1, 1)$.

(b) (8 pts) Find the work done by $\mathbf{F} = xy\mathbf{i} + y\mathbf{j} - yz\mathbf{k}$ over the curve $C : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} +$ $0 \le t \le 1$.

(c) (9 pts) Show that the differential form in the following integral is exact, then evaluate integral.

$$\int_{(1,1,0)}^{(5,0,8)} (e^y + yz)\,dx + (xe^y + xz)\,dy + (xy)\,dz$$

Problem 2 (answer on page 4 of the booklet.)

(10 pts) Suppose that the derivative of the function $f(x, y, z)$ at the point $P(1, 5, 2)$ is greates the direction of $\mathbf{A} = 3\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{k}$. Also suppose that the value of the derivative of f at P in direction of $\mathbf{B} = \mathbf{i} + \mathbf{j} - \sqrt{2}\mathbf{k}$ is 10. Find $\nabla f(P)$.

Problem 3 (answer on pages 5 and 6 of the booklet.)

(15 pts) Find the maximum value of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 4 (answer on pages 7, 8, and 9 of the booklet.)

Let D be the smaller region cut from the sphere $x^2 + y^2 + z^2 = 4$ by the plane $z = \sqrt{3}$.

(a) (6 pts) Set up the triple integral in spherical coordinates that give the volume of D using order of integration $d\rho\,d\phi\,d\theta$.

(b) (8 pts) Set up the triple integrals in spherical coordinates that give the volume of D using order of integration $d\phi\,d\rho\,d\theta$.

(c) (4 pts) Use the integral of part (a) to find the volume of D.

Problem 5 (answer on pages 10 and 11 of the booklet.)

(10 pts) Use the transformation

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2}, \qquad w = \frac{z}{3}$$

to rewrite

$$\int_0^3 \int_0^4 \int_{y/2}^{(y/2)+1} \left(\frac{2x - y}{2} + \frac{z}{3}\right) dx\,dy\,dz$$

as an integral over an appropriate region G in the uvw-space. Then evaluate the uvw-integral ove

Problem 6 (answer on pages 12 and 13 of the booklet.)

(4 pts each) Which of the following series converge, and which diverge? Find the sum of e convergent series.

$$(i)\ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n} + 5} \qquad (ii)\ \sum_{n=1}^{\infty} \frac{\ln(1 + e^n)}{n^2} \qquad (iii)\ \sum_{n=2}^{\infty} \frac{3^n}{n!}$$

Problem 7 (answer on pages 14 and 15 of the booklet.)

(a) (5 pts) Use that $(\arctan x)' = 1/(1 + x^2)$ to find a power series expansion for $\arctan x$ about point $x = 0$.

(b) (5 pts) For what values of x can we replace $\arctan x$ by $x - x^3/3$ with an error of magnitude greater than 2×10^{-16}?

(A) (1)

Math 201–Final Exam (Fall 04) B. Shayya

- Please write your **section number** on your booklet.
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- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)

(24 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(5,0,9)}^{(1,\pi,0)} (2x\cos y + yz)\, dx + (xz - x^2 \sin y)\, dy + (xy)\, dz$$

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Problem 2 (answer on pages **3** and **4** of the booklet.)

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(24 pts) Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 3 (answer on pages **5** and **6** of the booklet.)

Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$.

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(i) (8 pts) Set up the triple integrals in cylindrical coordinates that give the volume of D using the order of integration $dz\, r dr\, d\theta$. Then find the volume of D.

(ii) (6 pts) Set up the limits of integration for evaluating the integral of a function $f(x, y, z)$ over D as an iterated triple integral in the order $dy\, dz\, dx$.

(iii) (12 pts) Set up the triple integrals in spherical coordinates that give the volume of D using the order of integration $d\phi\, d\rho\, d\theta$.

Problem 4 (answer on pages **7** and **8** of the booklet.)

(25 pts) Integrate $g(x, y, z) = z$ over the surface of the prism cut from the first octant by the planes $z = x$, $z = 2 - x$, and $y = 2$.

Problem 5 (answer on pages **9**, **10**, and **11** of the booklet.)

Let S be the cone $z = 1 - \sqrt{x^2 + y^2}$, $0 \le z \le 1$, and let C be its base (i.e. C is the unit circle in the xy-plane). Find the counterclockwise circulation of the field

$$F(x, y, z) = x^2 y\mathbf{i} + 2y^3 z\mathbf{j} + 3z\mathbf{k}$$

around C

(a) (12 pts) directly,

(b) (8 pts) using Green's theorem, and

(c) (14 pts) using Stokes' theorem (i.e. by evaluating the flux of curl F outward through S).

Problem 6 (answer on pages **12** and **13** of the booklet.)

(25 pts) Let R be the region in the xy-plane bounded by the lines $y = 0$, $y = x$, $x + y = 4$, and $x + y = 9$. Use the transformation

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$$x = uv, \qquad y = (1 - u)v$$

to rewrite

$$\int\int_R \frac{1}{\sqrt{x+y}}\, dx\, dy$$

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- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(i) (8 pts) Find the line integral of $f(x,y,z) = xy + y + z$ over the straight-line segment from $(0,0,2)$ to $(2,1,0)$.

(ii) (17 pts) Let R be the region in the first quadrant bounded by the x-axis, the y-axis, and the line $x + y = 1$. Let C be the boundary of R traced counterclockwise. Find

$$\int_C y^2 dx + 3x^2 dy$$

(a) directly, and (b) using Green's theorem.

Problem 2 (answer on page 4 of the booklet.)

(8 pts) Find the surface area of the upper portion of the cylinder $x^2 + z^2 = 1$ that lies between the planes $x = \pm 1/2$ and $y = \pm 1/2$. (Hint. $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$.)

Problem 3 (answer on pages 5 and 6 of the booklet.)

(15 pts) Find the maximum and minimum values of $f(x,y,z) = x + 2y + 3z$ on the sphere $x^2 + y^2 + z^2 = 25$.

Problem 4 (answer on pages 7 and 8 of the booklet.)

Let D be the region that lies inside the sphere $\rho = 2$ and between the cones $\phi = \pi/6$ and $\phi = \pi/4$.

(i) (10 pts) Set up the triple integral in spherical coordinates that gives the volume of D. Then evaluate the integral.

(ii) (10 pts) Set up, but do not evaluate, the triple integrals in cylindrical coordinates that give the volume of D.

Problem 5 (answer on pages 9 and 10 of the booklet.)

(15 pts) Use the transformation

$$u = x, \qquad v = xy, \qquad w = 3z$$

to rewrite

$$\int_0^1 \int_1^2 \int_0^{2/x} (x^2 y + 3xyz)\, dy\, dx\, dz$$

as an integral over an appropriate region G in the uvw-space. Then evaluate the uvw-integral over G.

Problem 6 (answer on page 11 of the booklet.)

(8 pts) Use the fact that $(\arctan x)' = 1/(1+x^2)$ to find a power series expansion for $\arctan x$ about the point $x = 0$. Then decide whether $\sum_{n=1}^{\infty} n^{-0.1} \arctan(1/n)$ converges or diverges.

Problem 7 (answer on pages 12 and 13 of the booklet.)

(i) (5 pts) Find the Fourier series expansion of the function $f(x) = x$ over the interval $-\pi < x < \pi$. (Hint. Since f is an odd function, its Fourier series is of the form $\frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$.)

(ii) (2 pts) Find the sum of the series in part (i) for $-\pi \le x \le \pi$.

(iii) (2 pts) Use the result of part (ii) to show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

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- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages 1, 2, and 3 of the booklet.)

(i) (8 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(2,0,1)}^{(6,7,0)} \left(\sin(yz) - \frac{2xz}{1+x^2}\right) dx + xz\cos(yz)\, dy + \left(xy\cos(yz) - \ln(1+x^2)\right) dz$$

(ii) Find the flux of the field $F(x,y) = 2x\mathbf{i} - y\mathbf{j}$ across the region in the first quadrant bounded by the coordinate axes and the curve $y = x - x^2$, $0 \le x \le 1$,

(a) (8 pts) directly

(b) (6 pts) using Green's theorem.

(iii) (3 pts) Suppose $F(x,y)$ is a vector field on an open, connected, and simply connected region R in the xy-plane. Recall that the (two-dimensional) *curl test* says that

$$F \text{ is conservative in } R \Leftrightarrow \text{curl } F = 0 \text{ everywhere in } R.$$

Use Green's theorem to prove the $\Leftarrow$ implication.

Problem 2 (answer on pages 4, 5, and 6 of the booklet.)

(i) Let D be the region in space bounded below by the plane $z = 0$, above by the sphere $x^2+y^2+z^2 = 4$, and on the sides by the cylinder $x^2+y^2 = 1$. Set up (but do not evaluate) the iterated integrals in spherical coordinates that give the volume of D in the order

(a) (7 pts) $d\rho\, d\phi\, d\theta$

(b) (7 pts) $d\phi\, d\rho\, d\theta$.

(ii) (4 pts) Let D as in part (i). Set up (but do not evaluate) the iterated integral in cylindrical coordinates that give the volume of D in the order $dz\, dr\, d\theta$.

(iii) (8 pts) Let R be the region in space bounded by the planes $z = y$ and $y = 1$ and the surface $z = x^2$. Evaluate

$$\iiint_R \frac{2}{(x^2-1)^2}\, dV.$$

Problem 3 (answer on pages 7 and 8 of the booklet.)

(i) (8 pts) Find all local maxima, local minima, and saddle points of the function $f(x,y) = x^3 - y^3 - 2xy + 6$.

(ii) (7 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x,y) = 2x + 2y + 2z$ on the sphere $x^2+y^2+z^2 = 3$.

Problem 4 (answer on pages 9 and 10 of the booklet.)

(14 pts) Let R be the region in the xy-plane bounded by the lines $y = 0$, $y = 2x - 2$, $y = 4$, and $y = 2x$. Use the transformation

$$u = \frac{2x - y}{2}, \qquad v = \frac{y}{2}$$

to rewrite

$$\iint_R \frac{2x - y}{2}\, dA$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 5 (answer on page 11 of the booklet.)

(10 pts) Find the linearization $L(x, y)$ of the function $f(x, y) = x^2 - 3xy + 5$ at the point $(2, 1)$. Then estimate the error that results when one approximates $f(x, y)$ by $L(x, y)$ over the rectangle

$$R: \quad |x - 2| \le 0.1, \quad |y - 1| \le 0.1.$$

Problem 6 (answer on pages 12 and 13 of the booklet.)

(i) (3 pts) Does $\sum_{n=1}^{\infty} e^{1-n}$ converge? Why or why not?

(ii) (3 pts) What about $\sum_{n=1}^{\infty} e^{-1/n^2}$?

(iii) (4 pts) What about $\sum_{n=1}^{\infty} \left(1 - e^{-1/n^2}\right)$?

$\frac{u_{n+1}}{u_n}$

$\frac{e^{1-n+1}}{e^{1-n}} = \frac{e^{-n}}{e^{1-n}} = \frac{e^{-n}}{e \cdot e^{-n}} = \frac{1}{e} < 0$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 7 (answer on page **14** of the booklet.)

(6 pts each) Determine which of the following series converge, and which diverge.

(a) $\sum_{n=1}^{\infty} \sqrt{n} \ln\left(1 + \frac{1}{n^{2.1}}\right)$ (b) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^{1.2}}$ (c) $\sum_{n=1}^{\infty} n(\sqrt[n]{n} - 1)$

Problem 8 (answer on pages **15** and **16** of the booklet.)

(i) (6 pts) Use Taylor's theorem to prove that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad (-\infty < x < \infty).$$

(ii) (6 pts) Approximate

$$\int_0^{0.1} e^{-x^2} dx$$

with an error of magnitude less than 10^{-5}.

(iii) (6 pts) Show that

$$\int_0^{\infty} e^{-\pi x^2} dx = \frac{1}{2}.$$

(Hint. If $I = \int_0^{\infty} e^{-\pi x^2} dx$, then $I^2 = \int_0^{\infty} \int_0^{\infty} e^{-\pi(x^2+y^2)} dx dy$.)

(iv) (6 pts) Let E be the error resulting from the approximation

$$\int_0^{100} e^{-\pi x^2} dx \approx \frac{1}{2}.$$

Show that

$$|E| < \frac{e^{-5000\pi}}{2}.$$

Math 201–Final Exam (Spring 06)

- Please write your section number on your booklet.
- Please answer each problem on the indicated page(s) of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages 1 and 2 of the booklet.)

(a) (10 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(1,-2,5)}^{(2,-2,3)} (2xy - 3z^2)\,dx + (x^2 - 4yz)\,dy - 2(y^2 + 3xz)\,dz$$

(b) (10 pts) Find the counterclockwise circulation and outward flux of the field

$$F(x, y) = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$$

around and across the triangle bounded by the lines $y = 0$, $x = 3$, and $y = x$.

Problem 2 (answer on pages 3, 4, and 5 of the booklet.)

(a) (11 pts) Find the volume of the region in space that is bounded by the planes $x = 0$, $y = 0$, $z = x$, $x + z = 12$ and the paraboloid $y = x^2 + z^2$.

(b) (11 pts) Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

(c) (11 pts) Integrate $g(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Problem 3 (answer on pages 6 and 7 of the booklet.)

(12 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

Problem 4 (answer on pages 8 and 9 of the booklet.)

(10 pts) Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = 12x^2 + 12y^2 + (x + y)^3.$$

Problem 5 (answer on pages 10 and 11 of the booklet.)

(15 pts) Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation

$$x = \frac{u}{v}, \qquad y = uv$$

to rewrite

$$\int\int_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

Problem 6 (answer on pages 12 and 13 of the booklet.)

(a) (3 pts) Does $\sum_{n=1}^{\infty}(1 + \frac{1}{n^2})$ converge? Why or why not?

(b) (7 pts) What about $\sum_{n=1}^{\infty} \ln(1 + \frac{1}{n^2})$?

Math 201–Final Exam (Spring 07) B. Shayya

- Please write your **section number** on your booklet.
- Please answer each problem on the **indicated page(s)** of the booklet. Any part of your answer not written on the indicated page(s) will not be graded.
- Unjustified answers will receive little or no credit.

Problem 1 (answer on pages **1** and **2** of the booklet.)
(20 pts) Use the method of Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = e^{x+y+z}$ on the sphere $x^2 + y^2 + z^2 = 1$.

Problem 2 (answer on pages **3**, **4**, and **5** of the booklet.)
(i) (13 pts) Let V be the volume of the region in space that is bounded below by the xy-plane, on the sides by the cylinder $y = x^2$, and above by the plane $y + z = 1$. Write V as an iterated triple integral in the order $dydxdz$. Then find V.

(ii) Let D be the region in space bounded below by the plane $z = 1$, on the sides by the cylinder $x^2 + y^2 = 1$, and above by the sphere $x^2 + y^2 + z^2 = 4$. Suppose $f(x, y, z)$ is a continuous function on D and let

$$I = \iiint_D f(x, y, z)\,dV.$$

(a) (10 pts) Write I as an iterated triple integral(s) in cylindrical coordinates in the order $dzdrd\theta$.
(b) (13 pts) Write I as an iterated triple integral(s) in spherical coordinates in the order $d\rho d\phi d\theta$.
(c) (13 pts) Write I as an iterated triple integral(s) in spherical coordinates in the order $d\phi d\rho d\theta$.

Problem 3 (answer on pages **6**, **7**, and **8** of the booklet.)
(i) (18 pts) Show that the differential form in the following integral is exact, then evaluate the integral.

$$\int_{(1,-1,1)}^{(3,3,-1)} (3z^2 - 2xy)\,dx + (4yz - x^2)\,dy + (2y^2 + 6xz)\,dz$$

(ii) Let R be the region in the first quadrant that is bounded by the x-axis, the line $x = 1$, and the curve $y = x^2$. Find the outward flux of the field $F(x, y) = (x/2)\mathbf{i} + (y/2)\mathbf{j}$ across the boundary of R
(a) (13 pts) directly
(b) (8 pts) using Green's theorem.